Open-Loop and Closed-Loop Solvabilities for Stochastic Linear Quadratic Optimal Control Problems*

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August 11, 2015

Abstract: This paper is concerned with a stochastic linear quadratic (LQ, for short) optimal control problem. The notions of open-loop and closed-loop solvabilities are introduced. A simple example shows that these two solvabilities are different. Closed-loop solvability is established by means of solvability of the corresponding Riccati equation, which is implied by the uniform convexity of the quadratic cost functional. Conditions ensuring the convexity of the cost functional are discussed, including the issue that how negative the control weighting matrix-valued function $R(\cdot)$ can be. Finiteness of the LQ problem is characterized by the convergence of the solutions to a family of Riccati equations. Then, a minimizing sequence, whose convergence is equivalent to the open-loop solvability of the problem, is constructed. Finally, an illustrative example is presented.

Keywords: linear quadratic optimal control, stochastic differential equation, Riccati equation, finiteness, open-loop solvability, closed-loop solvability.

AMS Mathematics Subject Classification. 49N10, 49N35, 93E20.

1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space on which a standard one-dimensional Brownian motion $W = \{W(t); 0 \leqslant t < \infty\}$ is defined, where $\mathbb{F} = \{\mathcal{F}_t\}_{t\geqslant 0}$ is the natural filtration of W augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$ [16, 30]. Consider the following controlled linear stochastic differential equation (SDE, for short) on a finite horizon $[t, T]$:

$$\tag{1.1} \begin{cases} dX(s) = \big[A(s)X(s) + B(s)u(s) + b(s)\big]ds + \big[C(s)X(s) + D(s)u(s) + \sigma(s)\big]dW(s), \qquad s \in [t, T], \\ X(t) = x, \end{cases}$$

where $A(\cdot), B(\cdot), C(\cdot), D(\cdot)$ are given deterministic matrix-valued functions of proper dimensions, and $b(\cdot), \sigma(\cdot)$ are vector-valued $\mathbb{F}$-progressively measurable processes. In the above, $X(\cdot)$, valued in $\mathbb{R}^n$, is the *state process*, and $u(\cdot)$, valued in $\mathbb{R}^m$, is the *control process*. For any $t \in [0, T)$, we introduce the following Hilbert space:

$$\mathcal{U}[t, T] = \left\{ u : [t, T] \times \Omega \to \mathbb{R}^m \;\Big|\; u(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable}, \ \mathbb{E}\int_t^T |u(s)|^2 ds < \infty \right\}.$$

*This work is supported in part by NSF Grant DMS-1406776.
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Any $u(\cdot)\in\mathcal{U}[t,T]$ is called an *admissible control* (on $[t,T]$). Under some mild conditions on the coefficients, for any *initial pair* $(t,x)\in[0,T)\times\mathbb{R}^n$ and admissible control $u(\cdot)\in\mathcal{U}[t,T]$, (1.1) admits a unique strong solution $X(\cdot)\equiv X(\cdot\,;t,x,u(\cdot))$. Next we introduce the following cost functional:

$$
\begin{aligned}
J(t,x;u(\cdot))\triangleq\mathbb{E}\Bigg\{&\big\langle GX(T),X(T)\big\rangle+2\big\langle g,X(T)\big\rangle\\
&+\int_t^T\left[\left\langle\begin{pmatrix}Q(s)&S(s)^\top\\S(s)&R(s)\end{pmatrix}\begin{pmatrix}X(s)\\u(s)\end{pmatrix},\begin{pmatrix}X(s)\\u(s)\end{pmatrix}\right\rangle+2\left\langle\begin{pmatrix}q(s)\\\rho(s)\end{pmatrix},\begin{pmatrix}X(s)\\u(s)\end{pmatrix}\right\rangle\right]ds\Bigg\},
\end{aligned}
\tag{1.2}
$$

where G is a symmetric matrix, $Q(\cdot)$, $S(\cdot)$, $R(\cdot)$ are deterministic matrix-valued functions of proper dimensions with $Q(\cdot)^\top=Q(\cdot)$, $R(\cdot)^\top=R(\cdot)$; g is allowed to be an $\mathcal{F}_T$-measurable random variable and $q(\cdot),\rho(\cdot)$ are allowed to be vector-valued $\mathbb{F}$-progressively measurable processes. The classical stochastic LQ optimal control problem can be stated as follows.

Problem (SLQ). For any given initial pair $(t,x)\in[0,T)\times\mathbb{R}^n$, find a $u^*(\cdot)\in\mathcal{U}[t,T]$, such that

$$
J(t,x;u^*(\cdot))=\inf_{u(\cdot)\in\mathcal{U}[t,T]}J(t,x;u(\cdot))\triangleq V(t,x).
\tag{1.3}
$$

It is well-accepted that any $u^*(\cdot)\in\mathcal{U}[t,T]$ satisfying (1.3) is called an *optimal control* of Problem (SLQ) for the initial pair (t,x), and the corresponding $X^*(\cdot)\equiv X(\cdot\,;t,x,u^*(\cdot))$ is called an *optimal state process*; the pair $(X^*(\cdot),u^*(\cdot))$ is called an *optimal pair*. The function $V(\cdot\,,\cdot)$ is called the *value function* of Problem (SLQ). When $b(\cdot),\sigma(\cdot),g,q(\cdot),\rho(\cdot)=0$, we denote the corresponding Problem (SLQ) by Problem $(\text{SLQ})^0$. The corresponding cost functional and value function are denoted by $J^0(t,x;u(\cdot))$ and $V^0(t,x)$, respectively.

When the stochastic part is absent, with $b(\cdot)$, g, $q(\cdot)$ and $\rho(\cdot)$ being deterministic, Problem (SLQ) is reduced to a deterministic LQ optimal control problem, called Problem (DLQ). Hence, Problem (DLQ) can be regarded as a special case of Problem (SLQ). The history of Problem (DLQ) can be traced back to the work of Bellman–Glicksberg–Gross [5] in 1958, Kalman [15] in 1960 and Letov [18] in 1961. In the deterministic case, it is well known that $R(s)\geqslant 0$ is necessary for Problem (DLQ) to be finite (meaning that the infimum of the cost functional is finite). When the control weighting matrix $R(s)$ in the cost function is uniformly positive definite, under some mild additional conditions on the other weighting coefficients, the problem can be solved elegantly via the Riccati equation; see [4] for a thorough study of the Riccati equation approach (see also [30]). Stochastic LQ problems was firstly studied by Wonham [26] in 1968, followed by several researchers later (see [12, 6], for examples). In those works, the assumption that $R(s)>0$ was taken for granted. More precisely, under the following *standard* conditions:

$$
G\geqslant 0,\qquad R(s)\geqslant\delta I,\qquad Q(s)-S(s)^\top R(s)^{-1}S(s)\geqslant 0,\qquad \text{a.e.}\quad s\in[0,T],
\tag{1.4}
$$

for some $\delta>0$, the corresponding Riccati equation is uniquely solvable and Problem (SLQ) admits a unique optimal control which has a linear state feedback representation (see [30, Chapter 6] or [10]). In 1998, Chen–Li–Zhou [7] found that Problem (SLQ) might still be solvable even if $R(s)$ is not positive semi-definite. See also some follow-up works of Lim–Zhou [19], Chen–Zhou [10], and Chen–Yong [9], as well as the works of Hu–Zhou [13] and Qian–Zhou [23] on the study of solvability of indefinite Riccati equations (under certain technical conditions). In 2001, Ait Rami–Moore–Zhou [1] introduced a generalized Riccati equation involving the pseudo-inverse of a matrix and an additional algebraic constraint; see also Ait Rami–Zhou–Moore [2] for stochastic LQ optimal control problem on $[0,\infty)$, and a follow-up work of Wu–Zhou [27]. Recently, based on the work of Yong [28], Huang–Li–Yong [14] studied a mean-field LQ optimal control problem on $[0,\infty)$. For stochastic LQ optimal control problems with random coefficients, we further refer to the works of Chen–Yong [8], Tang [25], and Kohlmann–Tang [17].

Most recently, Sun–Yong [24] established that the existence of open-loop optimal controls is equivalent to the solvability of the corresponding optimality system which is a forward-backward stochastic differential equation (FBSDE, for short), and the existence of closed-loop optimal strategies is equivalent to the existence of a regular solution to the corresponding Riccati equation. From this point of view, this paper can be regarded as a continuation of [24], in a certain sense. Inspired by a result found in [29], we are able to cook up an example for which open-loop optimal controls exist, but the closed-loop optimal strategy does not exist. Because of this, it is necessary to distinguish open-loop and closed-loop solvabilities of Problem (SLQ). Next, having the equivalence between the solvability of the Riccati equation and the closed-loop solvability of Problem (SLQ), it is natural to seek conditions under which the Riccati equation is solvable, and the sought conditions are expected to be more general than (1.4) so that they could include some (although might not be all) cases that $R(\cdot)$ is allowed to be not positive semi-definite. One of our main results in this paper is to establish the equivalence between the *strongly regular solvability* of the Riccati equation (see below for a definition) and the uniform convexity of the cost functional. Note that uniform convexity condition is much weaker than (1.4), and is different from conditions imposed in [23].

Finiteness of Problem (SLQ) (meaning that the infimum of the cost functional is finite) is another important issue. The notion was introduced in [30] (see also [8, 9]). Some investigations were carried out in [21]. In this paper, due to the perfect structure of Problem $(\mathrm{SLQ})^0$, its finiteness will be characterized by the convergence of the solutions to a family of Riccati equations. As a by-product, we will construct minimizing sequences of Problem (SLQ) in a very natural way, and the convergence of the sequences will lead to the open-loop solvability of Problem (SLQ).

Among other things, we find several interesting facts which are listed below:

Fact 1. The value function $V(t,x)$ is not necessarily continuous in t even if Problem (SLQ) admits a *continuous* open-loop optimal control at all initial pair $(t,x)\in[0,T)\times\mathbb{R}^n$.

Fact 2. If Problem $(\mathrm{SLQ})^0$ is finite at t, then it is finite at any $t'>t$.

Fact 3. For Problem (SLQ) with $D(\cdot)=0$, under the assumption that $R(\cdot)$ is uniformly positive definite, without assuming the non-negativity of $Q(\cdot)$ and G, the finiteness and the unique closed-loop solvability of Problem (SLQ) are equivalent, which are also equivalent to the uniform convexity of the cost functional.

Fact 4. The existence of a regular solution to the Riccati equation implies the open-loop solvability of Problem (SLQ). However, it may happen that for any initial pair $(t,x)\in[0,T)\times\mathbb{R}^n$, Problem (SLQ) admits a *continuous* open-loop optimal control, while the Riccati equation does not admit a regular solution. This corrects an incorrect result found in [1] (see Section 4 for details).

The rest of the paper is organized as follows. Section 2 collects some preliminary results. In Section 3, we study the cost functional from a Hilbert space viewpoint and represent it as a quadratic functional of $u(\cdot)$. Section 4 is devoted to the strongly regular solvability of the Riccati equation under the uniform convexity of the cost functional. In Section 5, we discuss the finiteness of Problem (SLQ) as well as the convexity of the cost functional. In Section 6, we characterize the open-loop solvability of Problem (SLQ) by means of the convergence of minimizing sequences. An example is presented in section 7 to illustrate some relevant results obtained.

2 Preliminaries

We recall that $\mathbb{R}^n$ is the n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ is the space of all $(n\times m)$ matrices, endowed with the inner product $\langle M,N\rangle\mapsto \mathrm{tr}\,[M^\top N]$ and the norm $|M|=\sqrt{\mathrm{tr}\,[M^\top M]}$, $\mathbb{S}^n\subseteq\mathbb{R}^{n\times n}$ is the set of all

$(n\times n)$ symmetric matrices, $\overline{\mathbb{S}^n_+}\subseteq\mathbb{S}^n$ is the set of all $(n\times n)$ positive semi-definite matrices, and $\mathbb{S}^n_+\subseteq\overline{\mathbb{S}^n_+}$ is the set of all $(n\times n)$ positive-definite matrices. When there is no confusion, we shall use $\langle\cdot\,,\cdot\rangle$ for inner products in possibly different Hilbert spaces. Also, $M^\dagger$ stands for the (Moore-Penrose) pseudo-inverse of the matrix M ([22]), and $\mathcal{R}(M)$ stands for the range of the matrix M. Next, let $T>0$ be a fixed time horizon. For any $t\in[0,T)$ and Euclidean space $\mathbb{H}$, let

$$
\begin{aligned}
&C([t,T];\mathbb{H})=\Big\{\varphi:[t,T]\to\mathbb{H}\ \big|\ \varphi(\cdot)\text{ is continuous}\Big\},\\
&L^p(t,T;\mathbb{H})=\left\{\varphi:[t,T]\to\mathbb{H}\ \Big|\ \int_t^T|\varphi(s)|^pds<\infty\right\},\quad 1\leqslant p<\infty,\\
&L^\infty(t,T;\mathbb{H})=\left\{\varphi:[t,T]\to\mathbb{H}\ \Big|\ \operatorname*{esssup}_{s\in[t,T]}|\varphi(s)|<\infty\right\}.
\end{aligned}
$$

We denote

$$
\begin{aligned}
&L^2_{\mathcal{F}_T}(\Omega;\mathbb{H})=\Big\{\xi:\Omega\to\mathbb{H}\ \big|\ \xi\text{ is }\mathcal{F}_T\text{-measurable},\ \mathbb{E}|\xi|^2<\infty\Big\},\\
&L^2_{\mathbb{F}}(t,T;\mathbb{H})=\left\{\varphi:[t,T]\times\Omega\to\mathbb{H}\ \big|\ \varphi(\cdot)\text{ is }\mathbb{F}\text{-progressively measurable},\ \mathbb{E}\int_t^T|\varphi(s)|^2ds<\infty\right\},\\
&L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{H}))=\left\{\varphi:[t,T]\times\Omega\to\mathbb{H}\ \big|\ \varphi(\cdot)\text{ is }\mathbb{F}\text{-adapted, continuous},\ \mathbb{E}\left[\sup_{s\in[t,T]}|\varphi(s)|^2\right]<\infty\right\},\\
&L^2_{\mathbb{F}}(\Omega;L^1(t,T;\mathbb{H}))=\left\{\varphi:[t,T]\times\Omega\to\mathbb{H}\ \big|\ \varphi(\cdot)\text{ is }\mathbb{F}\text{-progressively measurable},\ \mathbb{E}\left(\int_t^T|\varphi(s)|ds\right)^2<\infty\right\}.
\end{aligned}
$$

For an $\mathbb{S}^n$-valued function $F(\cdot)$ on $[t,T]$, we use the notation $F(\cdot)\gg0$ to indicate that $F(\cdot)$ is uniformly positive definite on $[t,T]$, i.e., there exists a constant $\delta>0$ such that

$$F(s)\geqslant\delta I,\qquad\text{a.e. }\ s\in[t,T].$$

The following standard assumptions will be in force throughout this paper.

(H1) The coefficients of the state equation satisfy the following:

$$
\begin{cases}
A(\cdot)\in L^1(0,T;\mathbb{R}^{n\times n}),\quad B(\cdot)\in L^2(0,T;\mathbb{R}^{n\times m}),\quad b(\cdot)\in L^2_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^n)),\\
C(\cdot)\in L^2(0,T;\mathbb{R}^{n\times n}),\quad D(\cdot)\in L^\infty(0,T;\mathbb{R}^{n\times m}),\quad \sigma(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n).
\end{cases}
$$

(H2) The weighting coefficients in the cost functional satisfy the following:

$$
\begin{cases}
G\in\mathbb{S}^n,\quad Q(\cdot)\in L^1(0,T;\mathbb{S}^n),\quad S(\cdot)\in L^2(0,T;\mathbb{R}^{m\times n}),\quad R(\cdot)\in L^\infty(0,T;\mathbb{S}^m),\\
g\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n),\quad q(\cdot)\in L^2_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^n)),\quad \rho(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m).
\end{cases}
$$

By [24, Proposition 2.1], under (H1)–(H2), for any $(t,x)\in[0,T)\times\mathbb{R}^n$ and $u(\cdot)\in\mathcal{U}[t,T]$, the state equation (1.1) admits a unique strong solution $X(\cdot)\equiv X(\cdot\,;t,x,u(\cdot))$, and the cost functional (1.2) is well-defined. Then Problem (SLQ) makes sense. It is worthy of pointing out that in (H2), we do not impose any positive-definiteness/non-negativeness conditions on the weighting matrix/matrix-valued functions G, $Q(\cdot)$ and $R(\cdot)$. We now introduce the following definition.

Definition 2.1. (i) Problem (SLQ) is said to be *finite at initial pair* $(t,x)\in[0,T]\times\mathbb{R}^n$ if

$$V(t,x)>-\infty.\tag{2.1}$$

Problem (SLQ) is said to be *finite at* $t \in [0,T]$ if (2.1) holds for all $x \in \mathbb{R}^n$, and Problem (SLQ) is said to be *finite* if (2.1) holds for all $(t,x) \in [0,T] \times \mathbb{R}^n$.

(ii) An element $u^*(\cdot) \in \mathcal{U}[t,T]$ is called an *open-loop optimal control* of Problem (SLQ) for the initial pair $(t,x) \in [0,T] \times \mathbb{R}^n$ if

$$J(t,x;u^*(\cdot)) \leqslant J(t,x;u(\cdot)), \qquad \forall u(\cdot) \in \mathcal{U}[t,T]. \tag{2.2}$$

If an open-loop optimal control (uniquely) exists for $(t,x) \in [0,T] \times \mathbb{R}^n$, Problem (SLQ) is said to be (*uniquely*) *open-loop solvable at* $(t,x) \in [0,T] \times \mathbb{R}^n$. Problem (SLQ) is said to be (*uniquely*) *open-loop solvable at* $t \in [0,T)$ if for the given t, (2.2) holds for all $x \in \mathbb{R}^n$, and Problem (SLQ) is said to be (*uniquely*) *open-loop solvable* (on $[0,T) \times \mathbb{R}^n$) if it is (uniquely) open-loop solvable at all $(t,x) \in [0,T) \times \mathbb{R}^n$.

(iii) A pair $(\Theta^*(\cdot), v^*(\cdot)) \in L^2(t,T;\mathbb{R}^{m\times n}) \times \mathcal{U}[t,T]$ is called a *closed-loop optimal strategy* of Problem (SLQ) on $[t,T]$ if

$$J(t,x;\Theta^*(\cdot)X^*(\cdot)+v^*(\cdot)) \leqslant J(t,x;u(\cdot)), \qquad \forall x \in \mathbb{R}^n, \quad u(\cdot) \in \mathcal{U}[t,T], \tag{2.3}$$

where $X^*(\cdot)$ is the strong solution to the following closed-loop system:

$$\begin{cases} dX^*(s) = \Big\{\big[A(s)+B(s)\Theta^*(s)\big]X^*(s)+B(s)v^*(s)+b(s)\Big\}ds \\ \qquad\qquad +\Big\{\big[C(s)+D(s)\Theta^*(s)\big]X^*(s)+D(s)v^*(s)+\sigma(s)\Big\}dW(s), \qquad s \in [t,T], \\ X^*(t) = x. \end{cases} \tag{2.4}$$

If a closed-loop optimal strategy (uniquely) exists on $[t,T]$, Problem (SLQ) is said to be (*uniquely*) *closed-loop solvable on* $[t,T]$. Problem (SLQ) is said to be (*uniquely*) *closed-loop solvable* if it is (uniquely) closed-loop solvable on any $[t,T]$.

We emphasize that in the definition of closed-loop optimal strategy, (2.3) has to be true for all $x \in \mathbb{R}^n$. One sees that if $(\Theta^*(\cdot), v^*(\cdot))$ is a closed-loop optimal strategy of problem (SLQ) on $[t,T]$, then the outcome $u^*(\cdot) \equiv \Theta^*(\cdot)X^*(\cdot)+v^*(\cdot)$ is an open-loop optimal control of Problem (SLQ) for the initial pair $(t, X^*(t))$. Hence, the existence of closed-loop optimal strategies implies the existence of open-loop optimal controls. But, the existence of open-loop optimal controls does not necessarily imply the existence of a closed-loop optimal strategy. Here is such an example which is inspired by an example for deterministic LQ problems found in [29].

Example 2.2. Consider the following Problem $(\text{SLQ})^0$ with one-dimensional state equation:

$$\begin{cases} dX(s) = \big[u_1(s)+u_2(s)\big]ds + \big[u_1(s)-u_2(s)\big]dW(s), \qquad s \in [t,1], \\ X(t) = x, \end{cases} \tag{2.5}$$

and cost functional

$$J^0(t,x;u(\cdot)) = \mathbb{E}X(1)^2. \tag{2.6}$$

In this example, $u(\cdot) = (u_1(\cdot), u_2(\cdot))^\top$. It is clear that

$$V^0(t,x) = \inf_{u(\cdot)\in\mathcal{U}[t,T]} J^0(t,x;u(\cdot)) \geqslant 0, \qquad \forall (t,x) \in [0,1] \times \mathbb{R}.$$

On the other hand, for any $t \in [0,1)$, $\beta \geqslant \frac{1}{1-t}$, by taking

$$u^\beta(s) = -\frac{\beta x}{2}\mathbf{1}_{[t,t+\frac{1}{\beta}]}(s)\begin{pmatrix}1\\1\end{pmatrix}, \qquad s \in [t,1],$$

we have

$$X(s)=0,\qquad s\in[t+1/\beta,1].$$

Hence,

$$J^0(t,x;u^\beta(\cdot))=0,\qquad (t,x)\in[0,1)\times\mathbb{R}.$$

This implies that $\{u^\beta(\cdot)\mid\beta\geqslant\frac{1}{1-t}\}$ is a family of open-loop optimal controls of the corresponding Problem $(\text{SLQ})^0$ for the initial pair $(t,x)\in[0,1)\times\mathbb{R}$, and therefore,

$$V^0(t,x)=\begin{cases}0, & (t,x)\in[0,1)\times\mathbb{R},\\ x^2, & t=1,\ x\in\mathbb{R},\end{cases}$$

which is discontinuous at $t=1$, $x\neq 0$. Note also that if we take $\beta=\frac{1}{1-t}$, then the corresponding open-loop optimal control, denoted by $\bar u(\cdot)$, is given by

$$\bar u(s)=-\frac{x}{2(1-t)}\begin{pmatrix}1\\1\end{pmatrix},\qquad s\in[t,1],\tag{2.7}$$

which is a constant vector (only depends on the initial pair (t,x)). Thus, it is continuous (in $s\in[t,1]$). Now, we claim that this Problem $(\text{SLQ})^0$ is not closed-loop solvable on any $[t,1]$ with $t\in[0,1)$. In fact, if for some $t\in[0,1)$, there exists a closed-loop optimal strategy $(\Theta^*(\cdot),v^*(\cdot))$, then by definition, one must have

$$0\leqslant J^0(t,x;\Theta^*(\cdot)X^*(\cdot)+v^*(\cdot))\leqslant J^0(t,x;u^\beta(\cdot))=0,\qquad\forall x\in\mathbb{R}.\tag{2.8}$$

Let

$$\Theta^*(\cdot)=\begin{pmatrix}\Theta_1^*(\cdot)\\ \Theta_2^*(\cdot)\end{pmatrix},\qquad v^*(\cdot)=\begin{pmatrix}v_1^*(\cdot)\\ v_2^*(\cdot)\end{pmatrix}.$$

Then we have from (2.8) that the solution $X^*(\cdot)$ of the following closed-loop system:

$$\begin{cases}dX^*(s)=\Big\{\big[\Theta_1^*(s)+\Theta_2^*(s)\big]X^*(s)+\big[v_1^*(s)+v_2^*(s)\big]\Big\}ds\\ \qquad\qquad+\Big\{\big[\Theta_1^*(s)-\Theta_2^*(s)\big]X^*(s)+\big[v_1^*(s)-v_2^*(s)\big]\Big\}dW(s), & s\in[t,1],\\ X^*(t)=x,\end{cases}$$

must satisfy

$$X^*(1)=0,\qquad\forall x\in\mathbb{R}.$$

Note that

$$\begin{cases}d\big[\mathbb{E}X^*(s)\big]=\Big\{\big[\Theta_1^*(s)+\Theta_2^*(s)\big]\mathbb{E}X^*(s)+\mathbb{E}\big[v_1^*(s)+v_2^*(s)\big]\Big\}ds, & s\in[t,1],\\ \mathbb{E}X^*(t)=x.\end{cases}$$

Consequently,

$$0=\mathbb{E}X^*(1)=e^{\int_t^1[\Theta_1^*(s)+\Theta_2^*(s)]ds}x+\int_t^1 e^{\int_r^1[\Theta_1^*(s)+\Theta_2^*(s)]ds}\mathbb{E}[v_1^*(r)+v_2^*(r)]dr,\qquad\forall x\in\mathbb{R},$$

which is impossible since the above has to be true for all $x\in\mathbb{R}$. Hence, the corresponding Problem $(\text{SLQ})^0$ is not closed-loop solvable on any $[t,1]$ with $t\in[0,1)$, although the problem admits a continuous open-loop optimal control for any initial pair $(t,x)\in[0,1)\times\mathbb{R}^n$.

Due to the above indicated situation, unlike in [24], and in classical literature on LQ problems, we distinguish the notions of open-loop and closed-loop solvabilities for Problem (SLQ). We repeat here that for given initial time $t \in [0, T)$, an open-loop optimal control is allowed to depend on the initial state x, whereas, a closed-loop optimal strategy is required to be independent of the initial state x. Because of the nature of closed-loop strategies, we define the finiteness of Problem (SLQ) only in terms of open-loop controls.

To conclude this section, we present some lemmas which will be used frequently in sequel.

Lemma 2.3. *Let* (H1)–(H2) *hold and* $\Theta(\cdot) \in L^2(0, T; \mathbb{R}^{m\times n})$. *Let* $P(\cdot) \in C([0, T]; \mathbb{S}^n)$ *be the solution to the following Lyapunov equation:*

$$\begin{cases} \dot P + P(A + B\Theta) + (A + B\Theta)^\top P + (C + D\Theta)^\top P(C + D\Theta) \\ \quad + \Theta^\top R\Theta + S^\top \Theta + \Theta^\top S + Q = 0, \qquad \text{a.e. } \ s \in [0, T], \\ P(T) = G. \end{cases} \tag{2.9}$$

Then for any $(t, x) \in [0, T) \times \mathbb{R}^n$ *and* $u(\cdot) \in \mathcal{U}[t, T]$, *we have*

$$\begin{aligned} J^0(t, x; \Theta(\cdot)X(\cdot) + u(\cdot)) = \langle P(t)x, x\rangle + \mathbb{E}\int_t^T \Big\{ & 2\big\langle\big[B^\top P + D^\top PC + S + (R + D^\top PD)\Theta\big]X, u\big\rangle \\ & + \big\langle (R + D^\top PD)u, u\big\rangle\Big\} ds. \end{aligned}$$

Proof. For any $(t, x) \in [0, T) \times \mathbb{R}^n$ and $u(\cdot) \in \mathcal{U}[t, T]$, let $X(\cdot)$ be the solution of

$$\begin{cases} dX(s) = \big[(A + B\Theta)X + Bu\big]ds + \big[(C + D\Theta)X + Du\big]dW(s), \qquad s \in [t, T], \\ X(t) = x. \end{cases}$$

Applying Itô's formula to $s \mapsto \langle P(s)X(s), X(s)\rangle$, we have

$$\begin{aligned} & J^0(t, x; \Theta(\cdot)X(\cdot) + u(\cdot)) = \mathbb{E}\left\{\langle GX(T), X(T)\rangle + \int_t^T \left\langle \begin{pmatrix} Q & S^\top \\ S & R \end{pmatrix}\begin{pmatrix} X \\ \Theta X + u \end{pmatrix}, \begin{pmatrix} X \\ \Theta X + u \end{pmatrix}\right\rangle ds\right\} \\ & = \langle P(t)x, x\rangle + \mathbb{E}\int_t^T \Big\{\big\langle \dot P X, X\big\rangle + \big\langle P\big[(A + B\Theta)X + Bu\big], X\big\rangle + \big\langle PX, (A + B\Theta)X + Bu\big\rangle \\ & \qquad + \big\langle P\big[(C + D\Theta)X + Du\big], (C + D\Theta)X + Du\big\rangle + \big\langle QX, X\big\rangle \\ & \qquad + 2\big\langle SX, \Theta X + u\big\rangle + \big\langle R(\Theta X + u), \Theta X + u\big\rangle\Big\} ds \\ & = \langle P(t)x, x\rangle + \mathbb{E}\int_t^T \Big\{\big\langle\big[\dot P + P(A + B\Theta) + (A + B\Theta)^\top P + (C + D\Theta)^\top P(C + D\Theta) \\ & \qquad\quad + Q + \Theta^\top S + S^\top \Theta + \Theta^\top R\Theta\big]X, X\big\rangle \\ & \qquad + 2\big\langle\big[B^\top P + D^\top PC + S + (R + D^\top PD)\Theta\big]X, u\big\rangle + \big\langle (R + D^\top PD)u, u\big\rangle\Big\} ds \\ & = \langle P(t)x, x\rangle + \mathbb{E}\int_t^T \Big\{2\big\langle\big[B^\top P + D^\top PC + S + (R + D^\top PD)\Theta\big]X, u\big\rangle + \big\langle (R + D^\top PD)u, u\big\rangle\Big\} ds. \end{aligned}$$

This completes the proof. $\square$

The following lemma is concerned with the solution to a Lyapunov equation, whose proof can be found in [10] (see also [30, Chapter 6]).

Lemma 2.4. *Let*

$$\widetilde A(\cdot) \in L^1(0, T; \mathbb{R}^{n\times n}), \quad \widetilde C(\cdot) \in L^2(0, T; \mathbb{R}^{n\times n}), \quad \widetilde Q(\cdot) \in L^1(0, T; \mathbb{S}^n), \quad \widetilde G \in \mathbb{S}^n.$$

Then the following Lyapunov equation

$$\begin{cases} \dot{P}(s) + P(s)\widetilde{A}(s) + \widetilde{A}(s)^\top P(s) + \widetilde{C}(s)^\top P(s)\widetilde{C}(s) + \widetilde{Q}(s) = 0, \qquad \text{a.e.} \quad s \in [0,T], \\ P(T) = \widetilde{G}, \end{cases}$$

admits a unique solution $P(\cdot) \in C([0,T];\mathbb{S}^n)$ *given by*

$$P(t) = \mathbb{E}\left\{ \big[\widetilde{\Phi}(T)\widetilde{\Phi}(t)^{-1}\big]^\top \widetilde{G}\big[\widetilde{\Phi}(T)\widetilde{\Phi}(t)^{-1}\big] + \int_t^T \big[\widetilde{\Phi}(s)\widetilde{\Phi}(t)^{-1}\big]^\top \widetilde{Q}(s)\big[\widetilde{\Phi}(s)\widetilde{\Phi}(t)^{-1}\big] ds \right\},$$

where $\widetilde{\Phi}(\cdot)$ *is the solution of*

$$\begin{cases} d\widetilde{\Phi}(s) = \widetilde{A}(s)\widetilde{\Phi}(s)ds + \widetilde{C}(s)\widetilde{\Phi}(s)dW(s), \qquad s \geqslant 0, \\ \widetilde{\Phi}(0) = I. \end{cases}$$

Consequently, if

$$\widetilde{G} \geqslant 0, \qquad \widetilde{Q}(s) \geqslant 0, \qquad \text{a.e.} \quad s \in [0,T],$$

then $P(\cdot) \in C([0,T];\overline{\mathbb{S}^n_+})$.

Lemma 2.5. *For any* $u(\cdot) \in \mathcal{U}[t,T]$*, let* $X^{(u)}(\cdot)$ *be the solution of*

$$\begin{cases} dX^{(u)}(s) = \big[A(s)X^{(u)}(s) + B(s)u(s)\big]ds + \big[C(s)X^{(u)}(s) + D(s)u(s)\big]dW(s), \qquad s \in [t,T], \\ X^{(u)}(t) = 0. \end{cases} \tag{2.10}$$

Then for any $\Theta(\cdot) \in L^2(t,T;\mathbb{R}^{m\times n})$*, there exists a constant* $\gamma > 0$ *such that*

$$\mathbb{E}\int_t^T \big|u(s) - \Theta(s)X^{(u)}(s)\big|^2 ds \geqslant \gamma\mathbb{E}\int_t^T |u(s)|^2 ds, \qquad \forall u(\cdot) \in \mathcal{U}[t,T]. \tag{2.11}$$

Proof. Let $\Theta(\cdot) \in L^2(t,T;\mathbb{R}^{m\times n})$. Define a bounded linear operator $\mathfrak{L} : \mathcal{U}[t,T] \to \mathcal{U}[t,T]$ by

$$\mathfrak{L}u = u - \Theta X^{(u)}.$$

Then $\mathfrak{L}$ is bijective and its inverse $\mathfrak{L}^{-1}$ is given by

$$\mathfrak{L}^{-1}u = u + \Theta\widetilde{X}^{(u)},$$

where $\widetilde{X}^{(u)}(\cdot)$ is the solution of

$$\begin{cases} d\widetilde{X}^{(u)}(s) = \Big\{\big[A(s) + B(s)\Theta(s)\big]\widetilde{X}^{(u)}(s) + B(s)u(s)\Big\}ds \\ \qquad\qquad\qquad + \Big\{\big[C(s) + D(s)\Theta(s)\big]\widetilde{X}^{(u)}(s) + D(s)u(s)\Big\}dW(s), \qquad s \in [t,T], \\ \widetilde{X}^{(u)}(t) = 0. \end{cases}$$

By the bounded inverse theorem, $\mathfrak{L}^{-1}$ is bounded with norm $\|\mathfrak{L}^{-1}\| > 0$. Thus,

$$\begin{aligned} \mathbb{E}\int_t^T |u(s)|^2 ds &= \mathbb{E}\int_t^T |(\mathfrak{L}^{-1}\mathfrak{L}u)(s)|^2 ds \leqslant \|\mathfrak{L}^{-1}\|\mathbb{E}\int_t^T |(\mathfrak{L}u)(s)|^2 ds \\ &= \|\mathfrak{L}^{-1}\|\mathbb{E}\int_t^T \big|u(s) - \Theta(s)X^{(u)}(s)\big|^2 ds, \qquad \forall u(\cdot) \in \mathcal{U}[t,T], \end{aligned}$$

which implies (2.11) with $\gamma = \|\mathfrak{L}^{-1}\|^{-1}$. $\square$

Finally, we state the following extended Schur's lemma whose proof can be found in [3].

Lemma 2.6. *Let $Q \in \mathbb{S}^n$, $R \in \mathbb{S}^m$ and $S \in \mathbb{R}^{m\times n}$. Then*

$$\begin{pmatrix} Q & S^\top \\ S & R \end{pmatrix} \geqslant 0, \tag{2.12}$$

if and only if

$$Q - S^\top R^\dagger S \geqslant 0, \quad R \geqslant 0, \quad \mathcal{R}(S) \subseteq \mathcal{R}(R). \tag{2.13}$$

Note that the third condition in (2.13) is equivalent to the following:

$$S^\top (I - RR^\dagger) = 0. \tag{2.14}$$

3 Representation of the Cost Functional

In this section, we will present a representation of the cost functional for Problem (SLQ), from which we will obtain some basic conditions ensuring the convexity of the cost functional. Convexity of the cost functional will play a crucial role in the study of finiteness and open-loop/closed-loop solvability of Problem (SLQ). The following proposition is a summary of some relevant results found in [30].

Proposition 3.1. *Let* (H1)–(H2) *hold. For any $t \in [0,T)$, there exists a bounded self-adjoint linear operator $M_2(t) : \mathcal{U}[t,T] \to \mathcal{U}[t,T]$, a bounded linear operator $M_1(t) : \mathbb{R}^n \to \mathcal{U}[t,T]$, an $M_0(t) \in \mathbb{S}^n$ and $\nu_t(\cdot) \in \mathcal{U}[t,T], y_t \in \mathbb{R}^n, c_t \in \mathbb{R}$ such that*

$$\begin{aligned} &J(t,x;u(\cdot)) = \langle M_2(t)u,u\rangle + 2\langle M_1(t)x,u\rangle + \langle M_0(t)x,x\rangle + 2\langle u,\nu_t\rangle + 2\langle x,y_t\rangle + c_t, \\ &J^0(t,x;u(\cdot)) = \langle M_2(t)u,u\rangle + 2\langle M_1(t)x,u\rangle + \langle M_0(t)x,x\rangle, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall (x,u(\cdot)) \in \mathbb{R}^n \times \mathcal{U}[t,T]. \end{aligned} \tag{3.1}$$

Moreover, let $(X_0(\cdot), Y(\cdot), Z(\cdot))$ be the adapted solution of the following (decoupled) linear FBSDE:

$$\begin{cases} dX_0(s) = \big[A(s)X_0(s) + B(s)u(s)\big]ds + \big[C(s)X_0(s) + D(s)u(s)\big]dW(s), \qquad s \in [t,T], \\ dY(s) = -\big[A(s)^\top Y(s) + C(s)^\top Z(s) + Q(s)X_0(s) + S(s)^\top u(s)\big]ds + Z(s)dW(s), \qquad s \in [t,T], \\ X_0(t) = 0, \qquad Y(T) = GX_0(T). \end{cases} \tag{3.2}$$

Then

$$(M_2(t)u(\cdot))(s) = B(s)^\top Y(s) + D(s)^\top Z(s) + S(s)X_0(s) + R(s)u(s), \qquad s \in [t,T]. \tag{3.3}$$

Let $(\bar X_0(\cdot), \bar Y(\cdot), \bar Z(\cdot))$ be the adapted solution to the following (decoupled) FBSDE:

$$\begin{cases} d\bar X_0(s) = A(s)\bar X_0(s)ds + C(s)\bar X_0(s)dW(s), \qquad s \in [t,T], \\ d\bar Y(s) = -\big[A(s)^\top \bar Y(s) + C(s)^\top \bar Z(s) + Q(s)\bar X_0(s)\big]ds + \bar Z(s)dW(s), \qquad s \in [t,T], \\ \bar X_0(t) = x, \qquad \bar Y(T) = G\bar X_0(T). \end{cases} \tag{3.4}$$

Then

$$\begin{cases} (M_1(t)x)(s) = B(s)^\top \bar Y(s) + D(s)^\top \bar Z(s) + S(s)\bar X_0(s), \qquad s \in [t,T], \\ M_0(t)x = \mathbb{E}\big[\bar Y(t)\big]. \end{cases} \tag{3.5}$$

Also, let $(\widehat X_0(\cdot),\widehat Y(\cdot),\widehat Z(\cdot))$ be the adapted solution to the following (decoupled) FBSDE:

$$
\tag{3.6}
\begin{cases}
d\widehat X_0(s)=\big[A(s)\widehat X_0(s)+b(s)\big]ds+\big[C(s)\widehat X_0(s)+\sigma(s)\big]dW(s), \qquad s\in[t,T],\\
d\widehat Y(s)=-\big[A(s)^\top\widehat Y(s)+C(s)^\top\widehat Z(s)+Q(s)\widehat X_0(s)+q(s)\big]ds+\widehat Z(s)dW(s), \qquad s\in[t,T],\\
\widehat X_0(t)=0, \qquad \widehat Y(T)=G\widehat X_0(T)+g.
\end{cases}
$$

Then

$$
\tag{3.7}
\nu_t(s)=B(s)^\top\widehat Y(s)+D(s)^\top\widehat Z(s)+S(s)\widehat X_0(s)+\rho(s), \qquad s\in[t,T].
$$

Finally, $M_0(\cdot)$ solves the following Lyapunov equation:

$$
\tag{3.8}
\begin{cases}
\dot M_0(t)+M_0(t)A(t)+A(t)^\top M_0(t)+C(t)^\top M_0(t)C(t)+Q(t)=0, \qquad t\in[0,T],\\
M_0(T)=G,
\end{cases}
$$

and it admits the following representation:

$$
\tag{3.9}
M_0(t)=\mathbb{E}\left\{\big[\Phi(T)\Phi(t)^{-1}\big]^\top G\big[\Phi(T)\Phi(t)^{-1}\big]+\int_t^T\big[\Phi(s)\Phi(t)^{-1}\big]^\top Q(s)\big[\Phi(s)\Phi(t)^{-1}\big]ds\right\},
$$

where $\Phi(\cdot)$ is the solution to the following SDE for $\mathbb{R}^{n\times n}$-valued process:

$$
\tag{3.10}
\begin{cases}
d\Phi(s)=A(s)\Phi(s)ds+C(s)\Phi(s)dW(s), \qquad s\geqslant 0,\\
\Phi(0)=I.
\end{cases}
$$

Proof. Let $\Phi(\cdot)$ be the solution to (3.10). Then $\Phi(s)^{-1}$ exists for all $s\geqslant 0$ and the following holds:

$$
\begin{cases}
d\big[\Phi(s)^{-1}\big]=-\Phi(s)^{-1}\big[A(s)-C(s)^2\big]ds-\Phi(s)^{-1}C(s)dW(s), \qquad s\geqslant 0,\\
\Phi(0)^{-1}=I.
\end{cases}
$$

By the variation of constants formula, the solution $X(\cdot)\equiv X(\cdot\,;t,x,u(\cdot))$ of the state equation (1.1) can be written as follows:

$$
\begin{aligned}
X(s)=\Phi(s)\Phi(t)^{-1}x+\Phi(s)\left\{\int_t^s\Phi(r)^{-1}\big[B(r)-C(r)D(r)\big]u(r)dr+\int_t^s\Phi(r)^{-1}D(r)u(r)dW(r)\right\}\\
+\Phi(s)\left\{\int_t^s\Phi(r)^{-1}\big[b(r)-C(r)\sigma(r)\big]dr+\int_t^s\Phi(r)^{-1}\sigma(r)dW(r)\right\}, \qquad s\in[t,T].
\end{aligned}
$$

Now, let

$$
h_t(s)=\Phi(s)\left\{\int_t^s\Phi(r)^{-1}\big[b(r)-C(r)\sigma(r)\big]dr+\int_t^s\Phi(r)^{-1}\sigma(r)dW(r)\right\}, \qquad s\in[t,T],
$$

and define the following operators: For any $t\in[0,T)$, $(x,u(\cdot))\in\mathbb{R}^n\times\mathcal{U}[t,T]$,

$$
\begin{cases}
(L_tu)(\cdot)=\Phi(\cdot)\left\{\int_t^\cdot\Phi(r)^{-1}\big[B(r)-C(r)D(r)\big]u(r)dr+\int_t^\cdot\Phi(r)^{-1}D(r)u(r)dW(r)\right\},\\
(\Gamma_tx)(\cdot)=\Phi(\cdot)\Phi(t)^{-1}x, \quad \widehat L_tu=(L_tu)(T), \quad \widehat\Gamma_tx=(\Gamma_tx)(T).
\end{cases}
$$

Clearly, for any $t\in[0,T)$,

$$
L_t:\mathcal{U}[t,T]\to\mathcal{X}[t,T], \qquad \Gamma_t:\mathbb{R}^n\to\mathcal{X}[t,T], \qquad \widehat L_t:\mathcal{U}[t,T]\to\mathcal{X}_T, \qquad \widehat\Gamma_t:\mathbb{R}^n\to\mathcal{X}_T
$$

are all bounded linear operators, where $\mathcal{X}[t,T] \equiv L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$ and $\mathcal{X}_T \equiv L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$. Then, for any $t \in [0,T)$ and $(x,u(\cdot)) \in \mathbb{R}^n \times \mathcal{U}[t,T]$, the corresponding state process $X(\cdot)$ and its terminal value $X(T)$ are given by

$$\begin{cases} X(\cdot) = (\Gamma_t x)(\cdot) + (L_t u)(\cdot) + h_t(\cdot), \\ X(T) = \widehat{\Gamma}_t x + \widehat{L}_t u + h_t(T). \end{cases}$$

Hence, the cost functional can be written as

$$\begin{aligned} J(t,x;u(\cdot)) &= \big\langle G\big(\widehat{\Gamma}_t x + \widehat{L}_t u + h_t(T)\big), \widehat{\Gamma}_t x + \widehat{L}_t u + h_t(T)\big\rangle + 2\big\langle g, \widehat{\Gamma}_t x + \widehat{L}_t u + h_t(T)\big\rangle \\ &\quad + \big\langle Q(\Gamma_t x + L_t u + h_t), \Gamma_t x + L_t u + h_t\big\rangle + 2\big\langle S(\Gamma_t x + L_t u + h_t), u\big\rangle \\ &\quad + \langle Ru, u\rangle + 2\langle q, \Gamma_t x + L_t u + h_t\rangle + 2\langle \rho, u\rangle. \end{aligned}$$

In the above, $\langle\cdot\,,\cdot\rangle$ is used for inner products in possibly different spaces. Further, the adjoint operators

$$L_t^*: \mathcal{X}[t,T] \to \mathcal{U}[t,T], \qquad \Gamma_t^*: \mathcal{X}[t,T] \to \mathbb{R}^n, \qquad \widehat{L}_t^*: \mathcal{X}_T \to \mathcal{U}[t,T], \qquad \widehat{\Gamma}_t^*: \mathcal{X}_T \to \mathbb{R}^n$$

are given by the following:

$$(L_t^*\xi)(s) = B(s)^\top Y_0(s) + D(s)^\top Z_0(s), \quad s \in [t,T], \qquad \Gamma_t^*\xi = \mathbb{E}\big[Y_0(t)\big],$$

with $(Y_0(\cdot), Z_0(\cdot))$ being the adapted solution to the following backward stochastic differential equation (BSDE, for short):

$$\begin{cases} dY_0(s) = -\big[A(s)^\top Y_0(s) + C(s)^\top Z_0(s) + \xi(s)\big]ds + Z_0(s)dW(s), \qquad s \in [t,T], \\ Y_0(T) = 0, \end{cases} \tag{3.11}$$

and

$$(\widehat{L}_t^*\eta)(s) = B(s)^\top Y_1(s) + D(s)^\top Z_1(s), \quad s \in [t,T], \qquad \widehat{\Gamma}_t^*\eta = \mathbb{E}\big[Y_1(t)\big],$$

with $(Y_1(\cdot), Z_1(\cdot))$ being the adapted solution to the following BSDE:

$$\begin{cases} dY_1(s) = -\big[A(s)^\top Y_1(s) + C(s)^\top Z_1(s)\big]ds + Z_1(s)dW(s), \qquad s \in [t,T], \\ Y_1(T) = \eta \in \mathcal{X}_T. \end{cases} \tag{3.12}$$

Then with the well-defined adjoint operators, we can rewrite the cost functional as follows:

$$\begin{aligned} J(t,x;u(\cdot)) &= \big\langle\big(\widehat{L}_t^* G \widehat{L}_t + L_t^* Q L_t + S L_t + L_t^* S^\top + R\big)u, u\big\rangle \\ &\quad + 2\big\langle\big(\widehat{L}_t^* G \widehat{\Gamma}_t + L_t^* Q \Gamma_t + S\Gamma_t\big)x, u\big\rangle + \big\langle\big(\widehat{\Gamma}_t^* G \widehat{\Gamma}_t + \Gamma_t^* Q \Gamma_t\big)x, x\big\rangle \\ &\quad + 2\big\langle x, \widehat{\Gamma}_t^*\big[Gh_t(T) + g\big] + \Gamma_t^*\big(Qh_t + q\big)\big\rangle \\ &\quad + 2\big\langle u, \widehat{L}_t^*\big[Gh_t(T) + g\big] + L_t^*\big(Qh_t + q\big) + Sh_t + \rho\big\rangle \\ &\quad + \big\langle h_t(T), Gh_t(T) + 2g\big\rangle + \big\langle h_t, Qh_t + 2q\big\rangle \\ &\equiv \big\langle M_2(t)u, u\big\rangle + 2\big\langle M_1(t)x, u\big\rangle + \big\langle M_0(t)x, x\big\rangle + 2\big\langle x, y_t\big\rangle + 2\big\langle u, \nu_t\big\rangle + c_t, \end{aligned} \tag{3.13}$$

with $M_2(t): \mathcal{U}[t,T] \to \mathcal{U}[t,T]$ being bounded and self-adjoint, $M_1(t): \mathbb{R}^n \to \mathcal{U}[t,T]$ being bounded, and $M_0(t) \in \mathbb{S}^n$; $y_t \in \mathbb{R}^n$, $\nu_t(\cdot) \in \mathcal{U}[t,T]$ and $c_t \in \mathbb{R}$. Note that $\nu_t(\cdot), y_t, c_t = 0$ when $b(\cdot), \sigma(\cdot), g, q(\cdot), \rho(\cdot) = 0$. This gives (3.1). Further, if we let

$$X_0(\cdot) = (L_t u)(\cdot), \qquad Y(\cdot) = Y_0(\cdot) + Y_1(\cdot), \qquad Z(\cdot) = Z_0(\cdot) + Z_1(\cdot),$$

with

$$\xi(\cdot) = Q(\cdot)X_0(\cdot) + S(\cdot)^\top u(\cdot), \qquad \eta = GX_0(T),$$

then $X_0(\cdot)$ satisfies

$$\begin{cases} dX_0(s) = \big[A(s)X_0(s) + B(s)u(s)\big]ds + \big[C(s)X_0(s) + D(s)u(s)\big]dW(s), \qquad s \in [t,T], \\ X_0(t) = 0, \end{cases} \tag{3.14}$$

and $(Y(\cdot), Z(\cdot))$ is the adapted solution to the following BSDE:

$$\begin{cases} dY(s) = -\big[A(s)^\top Y(s) + C(s)^\top Z(s) + Q(s)X_0(s) + S(s)^\top u(s)\big]ds + Z(s)dW(s), \qquad s \in [t,T], \\ Y(T) = GX_0(T). \end{cases}$$

Thus (3.3) follows. If we let

$$\bar{X}_0(\cdot) = (\Gamma_t x)(\cdot), \qquad \bar{Y}(\cdot) = Y_0(\cdot) + Z_1(\cdot), \qquad \bar{Z}(\cdot) = Z_0(\cdot) + Z_1(\cdot),$$

with

$$\xi(\cdot) = Q(\cdot)\bar{X}_0(\cdot), \qquad \eta = G\bar{X}_0(T),$$

then $\bar{X}_0(\cdot)$ satisfies

$$\begin{cases} d\bar{X}_0(s) = A(s)\bar{X}_0(s)ds + C(s)\bar{X}_0(s)dW(s), \qquad s \in [t,T], \\ \bar{X}_0(t) = x, \end{cases} \tag{3.15}$$

and $(\bar{Y}(\cdot), \bar{Z}(\cdot))$ is the adapted solution to the following BSDE:

$$\begin{cases} d\bar{Y}(s) = -\big[A(s)^\top \bar{Y}(s) + C(s)^\top \bar{Z}(s) + Q(s)\bar{X}_0(s)\big]ds + \bar{Z}(s)dW(s), \qquad s \in [t,T], \\ \bar{Y}(T) = G\bar{X}_0(T). \end{cases}$$

Thus (3.5) follows. Likewise, if we let

$$\widehat{X}_0(\cdot) = h_t(\cdot), \qquad \widehat{Y}(\cdot) = Y_0(\cdot) + Z_1(\cdot), \qquad \widehat{Z}(\cdot) = Z_0(\cdot) + Z_1(\cdot),$$

with

$$\xi(\cdot) = Q(\cdot)\widehat{X}_0(\cdot) + q(\cdot), \qquad \eta = G\widehat{X}_0(T) + g,$$

then $\widehat{X}_0(\cdot)$ satisfies

$$\begin{cases} d\widehat{X}_0(s) = \big[A(s)\widehat{X}_0(s) + b(s)\big]ds + \big[C(s)\widehat{X}_0(s) + \sigma(s)\big]dW(s), \qquad s \in [t,T], \\ \widehat{X}_0(t) = 0, \end{cases} \tag{3.16}$$

and $(\widehat{Y}(\cdot), \widehat{Z}(\cdot))$ is the adapted solution to the following BSDE:

$$\begin{cases} d\widehat{Y}(s) = -\big[A(s)^\top \widehat{Y}(s) + C(s)^\top \widehat{Z}(s) + Q(s)\widehat{X}_0(s) + q(s)\big]ds + \widehat{Z}(s)dW(s), \qquad s \in [t,T], \\ \widehat{Y}(T) = G\widehat{X}_0(T) + g. \end{cases}$$

Thus (3.7) follows. Finally, we know that

$$\begin{aligned} \langle M_0(t)x, x\rangle = J^0(t,x;0) &= \mathbb{E}\left[\big\langle G\bar{X}_0(T), \bar{X}_0(T)\big\rangle + \int_t^T \big\langle Q(s)\bar{X}_0(s), \bar{X}_0(s)\big\rangle ds\right] \\ &= \mathbb{E}\left[\big\langle G\Phi(T)\Phi(t)^{-1}x, \Phi(T)\Phi(t)^{-1}x\big\rangle + \int_t^T \big\langle Q(s)\Phi(s)\Phi(t)^{-1}x, \Phi(s)\Phi(t)^{-1}x\big\rangle ds\right]. \end{aligned}$$

Then (3.9) follows. Also, by Lemma 2.4, we see that $M_0(\cdot)$ is the unique solution of Lyapunov equation (3.8). $\square$

From the representation of the cost functional, we have the following simple corollary.

Corollary 3.2. *Let* (H1)–(H2) *hold and* $t \in [0,T)$ *be given. For any* $x \in \mathbb{R}^n, \lambda \in \mathbb{R}$ *and* $u(\cdot), v(\cdot) \in \mathcal{U}[t,T]$*, the following holds:*

$$\begin{aligned}J(t,x;u(\cdot)+\lambda v(\cdot)) &= J(t,x;u(\cdot)) + \lambda^2 J^0(t,0;v(\cdot)) \\ &\qquad + 2\lambda\mathbb{E}\int_t^T \big\langle B(s)^\top Y(s)+D(s)^\top Z(s)+S(s)X(s)+R(s)u(s)+\rho(s), v(s)\big\rangle ds,\end{aligned} \tag{3.17}$$

where $(X(\cdot),Y(\cdot),Z(\cdot))$ *is the adapted solution to the following (decoupled) linear FBSDE:*

$$\begin{cases} dX(s) = \big[A(s)X(s)+B(s)u(s)+b(s)\big]ds \\ \qquad\qquad + \big[C(s)X(s)+D(s)u(s)+\sigma(s)\big]dW(s), \qquad s\in[t,T], \\ dY(s) = -\big[A(s)^\top Y(s)+C(s)^\top Z(s)+Q(s)X(s)+S(s)^\top u(s)+q(s)\big]ds \\ \qquad\qquad\qquad\qquad\qquad\qquad + Z(s)dW(s), \qquad s\in[t,T], \\ X(t)=x, \qquad Y(T)=GX(T)+g. \end{cases} \tag{3.18}$$

Consequently, the map $u(\cdot)\mapsto J(t,x;u(\cdot))$ *is Fréchet differentiable with the Fréchet derivative given by*

$$\mathcal{D}J(t,x;u(\cdot))(s) = 2\big[B(s)^\top Y(s)+D(s)^\top Z(s)+S(s)X(s)+R(s)u(s)+\rho(s)\big], \qquad s\in[t,T], \tag{3.19}$$

and (3.17) *can also be written as*

$$J(t,x;u(\cdot)+\lambda v(\cdot)) = J(t,x;u(\cdot)) + \lambda^2 J^0(t,0;v(\cdot)) + \lambda\mathbb{E}\int_t^T \big\langle \mathcal{D}J(t,x;u(\cdot))(s), v(s)\big\rangle ds. \tag{3.20}$$

Proof. From Proposition 3.1, we have

$$\begin{aligned}&J(t,x;u(\cdot)+\lambda v(\cdot)) \\ &= \big\langle M_2(t)(u+\lambda v), u+\lambda v\big\rangle + 2\big\langle M_1(t)x, u+\lambda v\big\rangle + \big\langle M_0(t)x,x\big\rangle + 2\big\langle u+\lambda v, \nu_t\big\rangle + 2\big\langle x, y_t\big\rangle + c_t \\ &= \big\langle M_2(t)u,u\big\rangle + 2\lambda\big\langle M_2(t)u,v\big\rangle + \lambda^2\big\langle M_2(t)v,v\big\rangle + 2\big\langle M_1(t)x,u\big\rangle + 2\lambda\big\langle M_1(t)x,v\big\rangle + \big\langle M_0(t)x,x\big\rangle \\ &\quad + 2\big\langle u,\nu_t\big\rangle + 2\lambda\big\langle v,\nu_t\big\rangle + 2\big\langle x,y_t\big\rangle + c_t \\ &= J(t,x;u(\cdot)) + \lambda^2 J^0(t,0;v(\cdot)) + 2\lambda\big\langle M_2(t)u + M_1(t)x + \nu_t, v\big\rangle.\end{aligned}$$

From the representation of $M_1(t)$, $M_2(t)$ and ν_t in Proposition 3.1, we see that

$$(M_2(t)u)(s) + (M_1(t)x)(s) + \nu_t(s) = B(s)^\top Y(s)+D(s)^\top Z(s)+S(s)X(s)+R(s)u(s)+\rho(s), \quad s\in[t,T],$$

with $(X(\cdot),Y(\cdot),Z(\cdot))$ being the adapted solution to the FBSDE (3.18). The rest of the proof is clear. $\square$

Note that if $u(\cdot)$ happens to be an open-loop optimal control of Problem (SLQ), then the following *stationarity condition* holds:

$$\mathcal{D}J(t,x;u(\cdot)) = 2\big[B(s)^\top Y(s)+D(s)^\top Z(s)+S(s)X(s)+R(s)u(s)+\rho(s)\big] = 0, \qquad s\in[t,T], \tag{3.21}$$

which brings a coupling into the FBSDE (3.18). We call (3.18), together with the stationarity condition (3.21), the *optimality system* for the open-loop optimal control of Problem (SLQ).

The following is concerned with the convexity of the cost functional, whose proof is straightforward, by making use of the representation (3.1) of the cost functional.

Corollary 3.3. *Let* (H1)–(H2) *hold and let* $t \in [0,T)$ *be given. Then the following are equivalent:*

(i) $u(\cdot) \mapsto J(t,x;u(\cdot))$ *is convex, for some* $x \in \mathbb{R}^n$.

(ii) $u(\cdot) \mapsto J(t,x;u(\cdot))$ *is convex, for any* $x \in \mathbb{R}^n$.

(iii) $u(\cdot) \mapsto J^0(t,x;u(\cdot))$ *is convex, for some* $x \in \mathbb{R}^n$.

(iv) $u(\cdot) \mapsto J^0(t,x;u(\cdot))$ *is convex, for any* $x \in \mathbb{R}^n$.

(v) $J^0(t,0;u(\cdot)) \geqslant 0$, *for all* $u(\cdot) \in \mathcal{U}[t,T]$.

(vi) $M_2(t) \geqslant 0$.

Similar to the above, we have that $u(\cdot) \mapsto J(t,x;u(\cdot))$ is uniformly convex if and only if

$$J^0(t,0;u(\cdot)) \geqslant \lambda \mathbb{E}\int_t^T |u(s)|^2 ds, \qquad \forall u(\cdot) \in \mathcal{U}[t,T], \tag{3.22}$$

for some $\lambda > 0$. This is also equivalent to the following:

$$M_2(t) \geqslant \lambda I, \tag{3.23}$$

for some $\lambda > 0$. Further, it is obvious that if the standard conditions (1.4) hold, then

$$M_2(t) = \widehat{L}_t^* G \widehat{L}_t + L_t^*(Q - S^\top R^{-1} S) L_t + (L_t^* S^\top R^{-\frac{1}{2}} + R^{\frac{1}{2}})(R^{-\frac{1}{2}} S L_t + R^{\frac{1}{2}}) \geqslant 0, \tag{3.24}$$

which means that the functional $u(\cdot) \mapsto J^0(t,0,u(\cdot))$ is convex. The following result tells us that under (1.4), one actually has the uniform convexity of the cost functional.

Proposition 3.4. *Let* (H1)–(H2) *and* (1.4) *hold. Then for any* $t \in [0,T)$, *the map* $u(\cdot) \mapsto J^0(t,0;u(\cdot))$ *is uniformly convex.*

Proof. For any $u(\cdot) \in \mathcal{U}[t,T]$, let $X^{(u)}(\cdot)$ be the solution of

$$\begin{cases} dX^{(u)}(s) = \big[A(s)X^{(u)}(s) + B(s)u(s)\big]ds + \big[C(s)X^{(u)}(s) + D(s)u(s)\big]dW(s), \qquad s \in [t,T], \\ X^{(u)}(t) = 0. \end{cases}$$

Then by Lemma 2.5 (taking $\Theta(\cdot) = -R(\cdot)^{-1}S(\cdot)$), we have

$$\begin{aligned} J^0(t,0;u(\cdot)) &= \mathbb{E}\left\{ \langle GX^{(u)}(T), X^{(u)}(T)\rangle + \int_t^T \Big[\langle QX^{(u)}, X^{(u)}\rangle + 2\langle SX^{(u)}, u\rangle + \langle Ru, u\rangle\Big] ds \right\} \\ &\geqslant \mathbb{E}\int_t^T \Big[\langle QX^{(u)}, X^{(u)}\rangle + 2\langle SX^{(u)}, u\rangle + \langle Ru, u\rangle\Big] ds \\ &= \mathbb{E}\int_t^T \Big[\big\langle (Q - S^\top R^{-1} S)X^{(u)}, X^{(u)}\big\rangle + \big\langle R\big(u + R^{-1}SX^{(u)}\big), u + R^{-1}SX^{(u)}\big\rangle\Big] ds \\ &\geqslant \delta \mathbb{E}\int_t^T \big|u + R^{-1}SX^{(u)}\big|^2 ds \geqslant \delta\gamma \mathbb{E}\int_t^T |u(s)|^2 ds, \qquad \forall u(\cdot) \in \mathcal{U}[t,T], \end{aligned}$$

for some $\gamma > 0$. This completes the proof. $\square$

4 Solvabilities of Problem (SLQ), Uniform Convexity of the Cost Functional, and the Riccati Equation

We begin with a simple result concerning the open-loop solvability of Problem (SLQ).

Proposition 4.1. *Let* (H1)–(H2) *hold. Suppose the map* $u(\cdot) \mapsto J^0(0,0;u(\cdot))$ *is uniformly convex. Then Problem* (SLQ) *is uniquely open-loop solvable, and there exists a constant* $\alpha \in \mathbb{R}$ *such that*

$$V^0(t,x) \geqslant \alpha |x|^2, \qquad \forall (t,x) \in [0,T] \times \mathbb{R}^n. \tag{4.1}$$

Note that in the above, the constant α does not have to be nonnegative.

Proof. First of all, by the uniform convexity of $u(\cdot) \mapsto J^0(0,0;u(\cdot))$, we may assume that

$$J^0(0,0;u(\cdot)) \geqslant \lambda \mathbb{E} \int_0^T |u(s)|^2 ds, \qquad \forall u(\cdot) \in \mathcal{U}[0,T], \tag{4.2}$$

for some $\lambda > 0$. Now, for any $t \in [0,T)$, and any $u(\cdot) \in \mathcal{U}[t,T]$, we define the *zero-extension* of $u(\cdot)$ as follows:

$$[\,0I_{[0,t)} \oplus u(\cdot)](s) = \begin{cases} 0, & s \in [0,t), \\ u(s), & s \in [t,T]. \end{cases} \tag{4.3}$$

Then $v(\cdot) \equiv 0I_{[0,t)} \oplus u(\cdot) \in \mathcal{U}[0,T]$, and due to the initial state being 0, the solution $X(s)$ of

$$\begin{cases} dX(s) = \big[A(s)X(s) + B(s)v(s)\big]ds + \big[C(s)X(s) + D(s)v(s)\big]dW(s), & s \in [0,T], \\ X(0) = 0, \end{cases}$$

satisfies

$$X(s) = 0, \qquad s \in [0,t].$$

Hence,

$$J^0(t,0;u(\cdot)) = J^0(0,0;0I_{[0,t)} \oplus u(\cdot)) \geqslant \lambda \mathbb{E} \int_0^T \big|[0I_{[0,t)} \oplus u(\cdot)](s)\big|^2 ds = \lambda \mathbb{E} \int_t^T |u(s)|^2 ds. \tag{4.4}$$

Thus, $u(\cdot) \mapsto J^0(t,x;u(\cdot))$ is uniformly convex for any given $(t,x) \in [0,T) \times \mathbb{R}^n$. By Corollary 3.2, we have

$$\begin{aligned} & J(t,x;u(\cdot)) = J(t,x;0) + J^0(t,0;u(\cdot)) + \mathbb{E} \int_t^T \big\langle \mathcal{D}J(t,x;0)(s), u(s) \big\rangle ds \\ & \geqslant J(t,x;0) + J^0(t,0;u(\cdot)) - \frac{\lambda}{2} \mathbb{E} \int_t^T |u(s)|^2 ds - \frac{1}{2\lambda} \mathbb{E} \int_t^T |\mathcal{D}J(t,x;0)(s)|^2 ds \\ & \geqslant \frac{\lambda}{2} \mathbb{E} \int_t^T |u(s)|^2 ds + J(t,x;0) - \frac{1}{2\lambda} \mathbb{E} \int_t^T |\mathcal{D}J(t,x;0)(s)|^2 ds, \qquad \forall u(\cdot) \in \mathcal{U}[t,T]. \end{aligned} \tag{4.5}$$

Consequently, by a standard argument involving minimizing sequence and locally weak compactness of Hilbert spaces, we see that for any given initial pair $(t,x) \in [0,T) \times \mathbb{R}^n$, Problem (SLQ) admits a unique open-loop optimal control. Moreover, when $b(\cdot), \sigma(\cdot), g, q(\cdot), \rho(\cdot) = 0$, (4.5) implies that

$$V^0(t,x) \geqslant J^0(t,x;0) - \frac{1}{2\lambda} \mathbb{E} \int_t^T |\mathcal{D}J^0(t,x;0)(s)|^2 ds. \tag{4.6}$$

Note that the functions on the right-hand side of (4.6) are quadratic in x and continuous in t. (4.1) follows immediately. $\square$

Now, we introduce the following Riccati equation associated with Problem (SLQ):

$$
\begin{cases}
\dot{P}(s)+P(s)A(s)+A(s)^\top P(s)+C(s)^\top P(s)C(s)+Q(s) \\
\quad -\big[P(s)B(s)+C(s)^\top P(s)D(s)+S(s)^\top\big]\big[R(s)+D(s)^\top P(s)D(s)\big]^\dagger \\
\qquad \cdot\big[B(s)^\top P(s)+D(s)^\top P(s)C(s)+S(s)\big]=0, \qquad \text{a.e. } \ s\in[0,T], \\
P(T)=G.
\end{cases} \tag{4.7}
$$

A solution $P(\cdot)\in C([0,T];\mathbb{S}^n)$ of (4.7) is said to be *regular* if

$$
\mathcal{R}\big(B(s)^\top P(s)+D(s)^\top P(s)C(s)+S(s)\big)\subseteq\mathcal{R}\big(R(s)+D(s)^\top P(s)D(s)\big), \qquad \text{a.e. } \ s\in[0,T], \tag{4.8}
$$

$$
\big[R(\cdot)+D(\cdot)^\top P(\cdot)D(\cdot)\big]^\dagger\big[B(\cdot)^\top P(\cdot)+D(\cdot)^\top P(\cdot)C(\cdot)+S(\cdot)\big]\in L^2(0,T;\mathbb{R}^{m\times n}), \tag{4.9}
$$

and

$$
R(s)+D(s)^\top P(s)D(s)\geqslant 0, \qquad \text{a.e. } \ s\in[0,T]. \tag{4.10}
$$

A solution $P(\cdot)$ of (4.7) is said to be *strongly regular* if

$$
R(s)+D(s)^\top P(s)D(s)\geqslant \lambda I, \qquad \text{a.e. } \ s\in[0,T], \tag{4.11}
$$

for some $\lambda>0$. The Riccati equation (4.7) is said to be (*strongly*) *regularly solvable*, if it admits a (strongly) regular solution. Clearly, condition (4.11) implies (4.8)–(4.10). Thus, a strongly regular solution $P(\cdot)$ must be regular. Moreover, it was shown in [24] that if a regular solution of (4.7) exists, it must be unique.

In [1], it was showed that for Problem $(\text{SLQ})^0$, the existence of a continuous open-loop optimal control for any initial pair $(t,x)\in[0,T]\times\mathbb{R}^n$ is equivalent to the solvability of the corresponding Riccati equation (4.7) with constraints (4.8) and (4.10). More precisely, their result can be stated as follows (in terms of our notations and equation numbers):

"***Theorem*** 4.2. *Suppose that* $B(\cdot),C(\cdot),D(\cdot),Q(\cdot),R(\cdot)$ *are continuous and* $S(\cdot)=0$. *Then Problem* $(\text{SLQ})^0$ *has a continuous open-loop optimal control for any initial pair* $(t,x)\in[0,T]\times\mathbb{R}^n$ *if and only if the Riccati equation* (4.7) *has a solution* $P(\cdot)$ *such that* (4.8) *and* (4.10) *hold.*"

This result is incorrect. Here is a simple counterexample.

Example 4.3. Consider the following deterministic one-dimensional state equation:

$$
\begin{cases}
dX(s)=u(s)ds, \qquad s\in[t,1], \\
X(t)=x,
\end{cases}
$$

and cost functional

$$
J(t,x;u(\cdot))=X(1)^2+\int_t^1 s^2u(s)^2ds.
$$

The Riccati equation for the above problem reads

$$
\begin{cases}
\dot{P}(t)=\dfrac{P(t)^2}{t^2}\mathbf{1}_{(0,1]}(t), \qquad \text{a.e. } \ t\in[0,1] \\
P(1)=1.
\end{cases} \tag{4.12}
$$

It is easy to see that $P(t)=t$ is the unique solution of (4.12), satisfying (4.8) and (4.10). Now, we claim that this problem does not admit an open-loop optimal control for initial pair $(0,x)$ with $x\neq 0$. In fact, if

for some $x \neq 0$, there exists an open-loop optimal control $u^*(\cdot) \in \mathcal{U}[0,T]$, then by the maximum principle, the solution $(X^*(\cdot), Y^*(\cdot))$ of the following (decoupled) forward-backward differential equation:

$$\begin{cases} \dot{X}^*(s) = u^*(s), \quad \dot{Y}^*(s) = 0, \qquad s \in [0,1], \\ X^*(0) = x, \quad Y^*(1) = X^*(1), \end{cases} \tag{4.13}$$

must satisfy

$$Y^*(s) + s^2 u^*(s) = 0, \qquad \text{a.e.} \quad s \in [0,1]. \tag{4.14}$$

Observe that the solution $(X^*(\cdot), Y^*(\cdot))$ of (4.13) is given by

$$X^*(s) = x + \int_0^s u^*(r)dr, \quad Y^*(s) \equiv X^*(1), \qquad s \in [0,1].$$

Hence,

$$u^*(s) = \frac{X^*(1)}{s^2}, \qquad \text{a.e.} \quad s \in (0,1].$$

Noting that $u^*(\cdot)$ is square-integrable, we must have $X^*(1) = 0$ and hence $u^*(\cdot) = 0$. Consequently,

$$x = X^*(1) - \int_0^1 u^*(r)dr = 0,$$

which is a contradiction.

From the above example, we see that the sufficiency part of the above Theorem 4.2 (a result from [1]) does not hold. We will see in Section 7 that the necessity part does not hold either.

Instead of Theorem 4.2, in [24], the following were proved, which establishes the equivalence between the closed-loop solvability of Problem (SLQ) and the regular solvability of the Riccati equation (4.7).

Theorem 4.4. *Let* (H1)–(H2) *hold. Problem* (SLQ) *is closed-loop solvable on* $[0,T]$ *if and only if the Riccati equation* (4.7) *admits a regular solution* $P(\cdot) \in C([0,T]; \mathbb{S}^n)$ *and the adapted solution* $(\eta(\cdot), \zeta(\cdot))$ *of the following BSDE:*

$$\begin{cases} d\eta(s) = -\Big\{\big[A^\top - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger B^\top\big]\eta \\ \qquad\qquad + \big[C^\top - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger D^\top\big]\zeta \\ \qquad\qquad + \big[C^\top - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger D^\top\big]P\sigma \\ \qquad\qquad - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger \rho + Pb + q\Big\}ds + \zeta dW(s), \quad s \in [0,T], \\ \eta(T) = g, \end{cases} \tag{4.15}$$

satisfies

$$\begin{cases} B^\top \eta + D^\top \zeta + D^\top P\sigma + \rho \in \mathcal{R}(R + D^\top PD), \qquad \text{a.e. a.s.} \\ (R + D^\top PD)^\dagger (B^\top \eta + D^\top \zeta + D^\top P\sigma + \rho) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m). \end{cases} \tag{4.16}$$

In this case, Problem (SLQ) *is closed-loop solvable on any* $[t,T]$*, and the closed-loop optimal strategy* $(\Theta^*(\cdot), v^*(\cdot))$ *admits the following representation:*

$$\begin{cases} \Theta^* = -(R + D^\top PD)^\dagger (B^\top P + D^\top PC + S) + \big[I - (R + D^\top PD)^\dagger (R + D^\top PD)\big]\Pi, \\ v^* = -(R + D^\top PD)^\dagger (B^\top \eta + D^\top \zeta + D^\top P\sigma + \rho) + \big[I - (R + D^\top PD)^\dagger (R + D^\top PD)\big]\nu, \end{cases} \tag{4.17}$$

for some $\Pi(\cdot)\in L^2(t,T;\mathbb{R}^{m\times n})$ *and* $\nu(\cdot)\in L^2_{\mathbb{F}}(t,T;\mathbb{R}^m)$, *and the value function is given by*

$$
\begin{aligned}
V(t,x)=\mathbb{E}\bigg\{&\langle P(t)x,x\rangle+2\langle\eta(t),x\rangle+\int_t^T\Big[\langle P\sigma,\sigma\rangle+2\langle\eta,b\rangle+2\langle\zeta,\sigma\rangle\\
&-\big\langle(R+D^\top PD)^\dagger(B^\top\eta+D^\top\zeta+D^\top P\sigma+\rho),B^\top\eta+D^\top\zeta+D^\top P\sigma+\rho\big\rangle\Big]ds\bigg\}.
\end{aligned}\tag{4.18}
$$

Note that in Example 4.3, the solution $P(t)=t$ to the Riccati equation (4.12) is not regular since

$$
[R(t)+D(t)^\top P(t)D(t)]^\dagger[B(t)^TP(t)+D(t)^\top P(t)C(t)+S(t)]=\frac{1}{t}\mathbf{1}_{(0,1]}(t)\notin L^2(0,1;\mathbb{R}).
$$

Hence, by Theorem 4.4, the corresponding LQ problem does not have a closed-loop optimal strategy.

From the above theorem, we see that the existence of a strongly regular solution to the Riccati equation (4.7) implies the unique closed-loop solvability of Problem (SLQ), which, by the remark right after Definition 2.1, implies the unique open-loop solvability of Problem (SLQ). Particularly, when $b(\cdot),\sigma(\cdot),g,q(\cdot),\rho(\cdot)=0$, the adapted solution of (4.15) is $(0,0)$, and (4.16) holds automatically. Thus, the existence of a regular solution to the Riccati equation (4.7) is equivalent to the closed-loop solvability of Problem (SLQ)^0, which implies the open-loop solvability of Problem (SLQ)^0. However, from Example 2.2, we know that the inverse is false, i.e., it may happen that for any initial pair $(t,x)\in[0,T)\times\mathbb{R}^n$, Problem (SLQ)^0 admits an open-loop optimal control (which could even be continuous), while the problem is not closed-loop solvable, which means that (4.7) does not admit a regular solution (see Section 7 for further details). On the other hand, it is known that under the standard conditions (1.4), the Riccati equation (4.7) admits a unique solution $P(\cdot)\in C([0,T];\overline{\mathbb{S}^n_+})$, and Problem (SLQ) admits a unique open-loop optimal control which has a state feedback form, represented via the solution of the Riccati equation (see [30, 10]). To summarize up, we have the following diagram:

$$
\begin{array}{ccccc}
 & \multicolumn{3}{c}{\boxed{G\geqslant 0,\quad R\gg 0,\quad Q-S^\top R^{-1}S\geqslant 0}} & \\
 & \Downarrow & & \Downarrow & \\
\boxed{u(\cdot)\mapsto J^0(t,x;u(\cdot))\text{ uniformly convex}} & & \boxed{\text{RE strongly regularly solvable}} & \Rightarrow & \boxed{\text{RE regularly solvable}}\\
\Downarrow & & \Downarrow & & \Updownarrow\\
\boxed{\text{(SLQ) uniquely open-loop solvable}} & \Leftarrow & \boxed{\text{(SLQ) uniquely closed-loop solvable}} & \Rightarrow & \boxed{\text{(SLQ)}^0\text{ closed-loop solvable}}
\end{array}
$$

where "RE" stands for the Riccati equation (4.7). It is clear that the uniform convexity of the map $u(\cdot)\mapsto J^0(t,x;u(\cdot))$ does not imply the standard conditions (1.4), which will be even clearer by the results of Section 5 below. Therefore, it is a desire to establish the following:

$$
\boxed{u(\cdot)\mapsto J^0(t,x;u(\cdot))\text{ uniformly convex}}\quad\Longleftrightarrow\quad\boxed{\text{RE strongly regularly solvable}}
$$

This is our next goal. To achieve this, we first present the following proposition, which plays a key technical role in this section.

Proposition 4.5. *Let* (H1)–(H2) *and* (4.2) *hold. Then for any* $\Theta(\cdot)\in L^2(0,T;\mathbb{R}^{m\times n})$, *the solution* $P(\cdot)\in C([0,T];\mathbb{S}^n)$ *to the following Lyapunov equation:*

$$
\begin{cases}
\dot P+P(A+B\Theta)+(A+B\Theta)^\top P+(C+D\Theta)^\top P(C+D\Theta)\\
\quad+\Theta^\top R\Theta+S^\top\Theta+\Theta^\top S+Q=0,\qquad\text{a.e. }\ s\in[0,T],\\
P(T)=G,
\end{cases}\tag{4.19}
$$

satisfies

$$R(t)+D(t)^\top P(t)D(t)\geqslant \lambda I,\quad \text{a.e. } t\in[0,T],\qquad \textit{and}\qquad P(t)\geqslant \alpha I,\quad \forall t\in[0,T], \tag{4.20}$$

where $\alpha\in\mathbb{R}$ *is the constant appears in* (4.1).

Proof. Let $\Theta(\cdot)\in L^2(0,T;\mathbb{R}^{m\times n})$ and let $P(\cdot)$ be the solution to (4.19). For any $u(\cdot)\in\mathcal{U}[0,T]$, let $X_0(\cdot)$ be the solution of

$$\begin{cases} dX_0(s)=\big[(A+B\Theta)X_0+Bu\big]ds+\big[(C+D\Theta)X_0+Du\big]dW(s), & s\in[0,T],\\ X_0(0)=0.\end{cases}$$

By (4.2) and Lemma 2.3, we have

$$\begin{aligned}&\lambda\mathbb{E}\int_0^T|\Theta(s)X_0(s)+u(s)|^2ds\leqslant J^0(0,0;\Theta(\cdot)X_0(\cdot)+u(\cdot))\\ &=\mathbb{E}\int_0^T\Big\{2\big\langle\big[B^\top P+D^\top PC+S+(R+D^\top PD)\Theta\big]X_0,u\big\rangle+\big\langle(R+D^\top PD)u,u\big\rangle\Big\}ds.\end{aligned}$$

Hence, for any $u(\cdot)\in\mathcal{U}[0,T]$, the following holds:

$$\begin{aligned}&\mathbb{E}\int_0^T\Big\{2\big\langle\big[B^\top P+D^\top PC+S+(R+D^\top PD-\lambda I)\Theta\big]X_0,u\big\rangle\\ &\qquad+\big\langle(R+D^\top PD-\lambda I)u,u\big\rangle\Big\}ds=\lambda\mathbb{E}\int_0^T|\Theta(s)X_0(s)|^2ds\geqslant 0.\end{aligned}\tag{4.21}$$

Now, fix any $u_0\in\mathbb{R}^m$, take $u(s)=u_0\mathbf{1}_{[t,t+h]}(s)$, with $0\leqslant t<t+h\leqslant T$. Then

$$\begin{cases} d\big[\mathbb{E}X_0(s)\big]=\Big\{\big[A(s)+B(s)\Theta(s)\big]\mathbb{E}X_0(s)+B(s)u_0\mathbf{1}_{[t,t+h]}(s)\Big\}ds, & s\in[0,T],\\ \mathbb{E}X_0(0)=0.\end{cases}$$

Hence,

$$\mathbb{E}X_0(s)=\begin{cases}0, & s\in[0,t],\\ \Phi(s)\displaystyle\int_t^{s\wedge(t+h)}\Phi(r)^{-1}B(r)u_0dr, & s\in[t,T],\end{cases}$$

where $\Phi(\cdot)$ is the solution of the following $\mathbb{R}^{n\times n}$-valued ordinary differential equation:

$$\begin{cases}\dot\Phi(s)=\big[A(s)+B(s)\Theta(s)\big]\Phi(s), & s\in[0,T],\\ \Phi(0)=I.\end{cases}$$

Consequently, (4.21) becomes

$$\begin{aligned}&\int_t^{t+h}\Big\{2\Big\langle\big[B^\top P+D^\top PC+S+(R+D^\top PD-\lambda I)\Theta\big]\Phi(s)\int_t^s\Phi(r)^{-1}B(r)u_0dr,u_0\Big\rangle\\ &\qquad+\big\langle(R+D^\top PD-\lambda I)u_0,u_0\big\rangle\Big\}ds\geqslant 0.\end{aligned}$$

Dividing both sides of the above by h and letting $h\to 0$, we obtain

$$\big\langle\big[R(t)+D(t)^\top P(t)D(t)-\lambda I\big]u_0,u_0\big\rangle\geqslant 0,\qquad \text{a.e. } t\in[0,T],\quad \forall u_0\in\mathbb{R}^m.$$

The first inequality in (4.20) follows. To prove the second, for any $(t,x)\in[0,T)\times\mathbb{R}^n$ and $u(\cdot)\in\mathcal{U}[t,T]$, let $X(\cdot)$ be the solution of

$$\begin{cases} dX(s)=\big[(A+B\Theta)X+Bu\big]ds+\big[(C+D\Theta)X+Du\big]dW(s), \qquad s\in[t,T],\\ X(t)=x.\end{cases}$$

By Proposition 4.1 and Lemma 2.3, we have

$$\begin{aligned}\alpha|x|^2&\leqslant V^0(t,x)\leqslant J^0(t,x;\Theta(\cdot)X(\cdot)+u(\cdot))\\ &=\langle P(t)x,x\rangle+\mathbb{E}\int_t^T\Big\{2\big\langle\big[B^\top P+D^\top PC+S+(R+D^\top PD)\Theta\big]X,u\big\rangle+\big\langle(R+D^\top PD)u,u\big\rangle\Big\}ds.\end{aligned}$$

In particular, by taking $u(\cdot)=0$ in the above, we obtain

$$\langle P(t)x,x\rangle\geqslant\alpha|x|^2,\qquad\forall(t,x)\in[0,T]\times\mathbb{R}^n,$$

and the second inequality therefore follows. $\square$

Now we state the main result of this section.

Theorem 4.6. *Let* (H1)–(H2) *hold. Then the following statements are equivalent:*

(i) *The map $u(\cdot)\mapsto J^0(0,0;u(\cdot))$ is uniformly convex, i.e., there exists a $\lambda>0$ such that* (4.2) *holds.*

(ii) *The Riccati equation* (4.7) *admits a strongly regular solution $P(\cdot)\in C([0,T];\mathbb{S}^n)$.*

Proof. (i) $\Rightarrow$ (ii). Let P_0 be the solution of

$$\begin{cases}\dot P_0+P_0A+A^\top P_0+C^\top P_0C+Q=0, \qquad \text{a.e. } s\in[0,T],\\ P_0(T)=G.\end{cases}\tag{4.22}$$

Applying Proposition 4.5 with $\Theta=0$, we obtain that

$$R(t)+D(t)^\top P_0(t)D(t)\geqslant\lambda I,\quad P_0(t)\geqslant\alpha I,\qquad\text{a.e. } t\in[0,T].$$

Next, inductively, for $i=0,1,2,\cdots$, we set

$$\Theta_i=-(R+D^\top P_iD)^{-1}(B^\top P_i+D^\top P_iC+S),\qquad A_i=A+B\Theta_i,\qquad C_i=C+D\Theta_i,\tag{4.23}$$

and let P_{i+1} be the solution of

$$\begin{cases}\dot P_{i+1}+P_{i+1}A_i+A_i^\top P_{i+1}+C_i^\top P_{i+1}C_i+\Theta_i^\top R\Theta_i+S^\top\Theta_i+\Theta_i^\top S+Q=0, \qquad \text{a.e. } s\in[0,T],\\ P_{i+1}(T)=G.\end{cases}\tag{4.24}$$

By Proposition 4.5, we see that

$$R(t)+D(t)^\top P_{i+1}(t)D(t)\geqslant\lambda I,\quad P_{i+1}(t)\geqslant\alpha I,\qquad\text{a.e. } s\in[0,T],\qquad i=0,1,2,\cdots.\tag{4.25}$$

We now claim that $\{P_i\}_{i=1}^\infty$ converges uniformly in $C([0,T];\mathbb{S}^n)$. To show this, let

$$\Delta_i\triangleq P_i-P_{i+1},\qquad\Lambda_i\triangleq\Theta_{i-1}-\Theta_i,\qquad i\geqslant1.$$

Then for $i \geqslant 1$, we have

$$
\begin{aligned}
-\dot{\Delta}_i &= \dot{P}_{i+1} - \dot{P}_i \\
&= P_iA_{i-1} + A_{i-1}^\top P_i + C_{i-1}^\top P_iC_{i-1} + \Theta_{i-1}^\top R\Theta_{i-1} + S^\top\Theta_{i-1} + \Theta_{i-1}^\top S \\
&\quad - P_{i+1}A_i - A_i^\top P_{i+1} - C_i^\top P_{i+1}C_i - \Theta_i^\top R\Theta_i - S^\top\Theta_i - \Theta_i^\top S \\
&= \Delta_iA_i + A_i^\top\Delta_i + C_i^\top\Delta_iC_i + P_i(A_{i-1} - A_i) + (A_{i-1} - A_i)^\top P_i \\
&\quad + C_{i-1}^\top P_iC_{i-1} - C_i^\top P_iC_i + \Theta_{i-1}^\top R\Theta_{i-1} - \Theta_i^\top R\Theta_i + S^\top\Lambda_i + \Lambda_i^\top S.
\end{aligned} \tag{4.26}
$$

By (4.23), we have the following:

$$
\begin{cases}
A_{i-1} - A_i = B\Lambda_i, \qquad C_{i-1} - C_i = D\Lambda_i, \\
C_{i-1}^\top P_iC_{i-1} - C_i^\top P_iC_i = \Lambda_i^\top D^\top P_iD\Lambda_i + C_i^\top P_iD\Lambda_i + \Lambda_i^\top D^\top P_iC_i, \\
\Theta_{i-1}^\top R\Theta_{i-1} - \Theta_i^\top R\Theta_i = \Lambda_i^\top R\Lambda_i + \Lambda_i^\top R\Theta_i + \Theta_i^\top R\Lambda_i.
\end{cases} \tag{4.27}
$$

Note that

$$
B^\top P_i + D^\top P_iC_i + R\Theta_i + S = B^\top P_i + D^\top P_iC + S + (R + D^\top P_iD)\Theta_i = 0.
$$

Thus, plugging (4.27) into (4.26) yields

$$
\begin{aligned}
&-(\dot{\Delta}_i + \Delta_iA_i + A_i^\top\Delta_i + C_i^\top\Delta_iC_i) \\
&= P_iB\Lambda_i + \Lambda_i^\top B^\top P_i + \Lambda_i^\top D^\top P_iD\Lambda_i + C_i^\top P_iD\Lambda_i + \Lambda_i^\top D^\top P_iC_i \\
&\quad + \Lambda_i^\top R\Lambda_i + \Lambda_i^\top R\Theta_i + \Theta_i^\top R\Lambda_i + S^\top\Lambda_i + \Lambda_i^\top S \\
&= \Lambda_i^\top(R + D^\top P_iD)\Lambda_i + (P_iB + C_i^\top P_iD + \Theta_i^\top R + S^\top)\Lambda_i + \Lambda_i^\top(B^\top P_i + D^\top P_iC_i + R\Theta_i + S) \\
&= \Lambda_i^\top(R + D^\top P_iD)\Lambda_i \geqslant 0.
\end{aligned} \tag{4.28}
$$

Noting that $\Delta_i(T) = 0$ and using Lemma 2.4, also noting (4.25), we obtain

$$
P_1(s) \geqslant P_i(s) \geqslant P_{i+1}(s) \geqslant \alpha I, \qquad \forall s \in [0,T], \quad \forall i \geqslant 1.
$$

Therefore, the sequence $\{P_i\}_{i=1}^\infty$ is uniformly bounded. Consequently, there exists a constant $K > 0$ such that (noting (4.25))

$$
\begin{cases}
|P_i(s)|, \ |R_i(s)| \leqslant K, \\
|\Theta_i(s)| \leqslant K\big(|B(s)| + |C(s)| + |S(s)|\big), \\
|A_i(s)| \leqslant |A(s)| + K|B(s)|\big(|B(s)| + |C(s)| + |S(s)|\big), \\
|C_i(s)| \leqslant |C(s)| + K\big(|B(s)| + |C(s)| + |S(s)|\big),
\end{cases} \qquad \text{a.e.} \quad s \in [0,T], \quad \forall\ i \geqslant 0, \tag{4.29}
$$

where $R_i(s) \triangleq R(s) + D^\top(s)P_i(s)D(s)$. Observe that

$$
\begin{aligned}
\Lambda_i &= \Theta_{i-1} - \Theta_i \\
&= R_i^{-1}D^\top\Delta_{i-1}DR_{i-1}^{-1}(B^\top P_i + D^\top P_iC + S) - R_{i-1}^{-1}(B^\top\Delta_{i-1} + D^\top\Delta_{i-1}C).
\end{aligned} \tag{4.30}
$$

Thus, noting (4.29), one has

$$
\begin{aligned}
|\Lambda_i(s)^\top R_i(s)\Lambda_i(s)| &\leqslant \Big(|\Theta_i(s)| + |\Theta_{i-1}(s)|\Big)\,|R_i(s)|\,|\Theta_{i-1}(s) - \Theta_i(s)| \\
&\leqslant K\Big(|B(s)| + |C(s)| + |S(s)|\Big)^2|\Delta_{i-1}(s)|.
\end{aligned} \tag{4.31}
$$

Equation (4.28), together with $\Delta_i(T)=0$, implies that

$$\Delta_i(s)=\int_s^T \big(\Delta_i A_i+A_i^\top\Delta_i+C_i^\top\Delta_i C_i+\Lambda_i^\top R_i\Lambda_i\big)dr.$$

Making use of (4.31) and still noting (4.29), we get

$$|\Delta_i(s)|\leqslant\int_s^T \varphi(r)\Big[|\Delta_i(r)|+|\Delta_{i-1}(r)|\Big]dr,\qquad \forall s\in[0,T],\quad \forall i\geqslant 1,$$

where $\varphi(\cdot)$ is a nonnegative integrable function independent of $\Delta_i(\cdot)$. By Gronwall's inequality,

$$|\Delta_i(s)|\leqslant e^{\int_0^T\varphi(r)dr}\int_s^T \varphi(r)|\Delta_{i-1}(r)|dr\equiv c\int_s^T \varphi(r)|\Delta_{i-1}(r)|dr.$$

Set

$$a\triangleq\max_{0\leqslant s\leqslant T}|\Delta_0(s)|.$$

By induction, we deduce that

$$|\Delta_i(s)|\leqslant a\frac{c^i}{i!}\Big(\int_s^T\varphi(r)dr\Big)^i,\qquad \forall s\in[0,T],$$

which implies the uniform convergence of $\{P_i\}_{i=1}^\infty$. We denote P the limit of $\{P_i\}_{i=1}^\infty$, then (noting (4.25))

$$R(s)+D(s)^\top P(s)D(s)=\lim_{i\to\infty}R(s)+D(s)^\top P_i(s)D(s)\geqslant\lambda I,\qquad \text{a.e. }\ s\in[0,T],$$

and as $i\to\infty$,

$$\begin{cases}\Theta_i\to-(R+D^\top PD)^{-1}(B^\top P+D^\top PC+S)\equiv\Theta\quad\text{in }L^2,\\ A_i\to A+B\Theta\quad\text{in }L^1,\qquad C_i\to C+D\Theta\quad\text{in }L^2.\end{cases}$$

Therefore, $P(\cdot)$ satisfies the following equation:

$$\begin{cases}\dot P+P(A+B\Theta)+(A+B\Theta)^\top P+(C+D\Theta)^\top P(C+D\Theta)\\ \quad+\Theta^\top R\Theta+S^\top\Theta+\Theta^\top S+Q=0,\qquad \text{a.e. }\ s\in[0,T],\\ P(T)=G,\end{cases}$$

which is equivalent to (4.7).

(ii) $\Rightarrow$ (i). Let $P(\cdot)$ be the strongly regular solution of (4.7). Then there exists a $\lambda>0$ such that

$$R(s)+D(s)^\top P(s)D(s)\geqslant\lambda I,\qquad \text{a.e. }\ s\in[0,T]. \tag{4.32}$$

Set

$$\Theta\triangleq-(R+D^\top PD)^{-1}(B^\top P+D^\top PC+S)\in L^2(0,T;\mathbb{R}^{m\times n}).$$

For any $u(\cdot)\in\mathcal{U}[0,T]$, let $X^{(u)}(\cdot)$ be the solution of

$$\begin{cases}dX^{(u)}(s)=\big[A(s)X^{(u)}(s)+B(s)u(s)\big]ds+\big[C(s)X^{(u)}(s)+D(s)u(s)\big]dW(s),\qquad s\in[0,T],\\ X(0)=0.\end{cases}$$

Applying Itô's formula to $s \mapsto \langle P(s)X^{(u)}(s), X^{(u)}(s)\rangle$, we have

$$
\begin{aligned}
J^0(0,0;u(\cdot)) &= \mathbb{E}\left\{\big\langle GX^{(u)}(T), X^{(u)}(T)\big\rangle + \int_0^T \left\langle \begin{pmatrix} Q & S^\top \\ S & R \end{pmatrix}\begin{pmatrix} X^{(u)} \\ u \end{pmatrix}, \begin{pmatrix} X^{(u)} \\ u \end{pmatrix}\right\rangle ds\right\} \\
&= \mathbb{E}\int_0^T \Big[\big\langle \dot{P}X^{(u)}, X^{(u)}\big\rangle + \big\langle P\big(AX^{(u)}+Bu\big), X^{(u)}\big\rangle + \big\langle PX^{(u)}, AX^{(u)}+Bu\big\rangle \\
&\qquad + \big\langle P\big(CX^{(u)}+Du\big), CX^{(u)}+Du\big\rangle + \big\langle QX^{(u)}, X^{(u)}\big\rangle + 2\big\langle SX^{(u)}, u\big\rangle + \big\langle Ru, u\big\rangle\Big] ds \\
&= \mathbb{E}\int_0^T \Big[\big\langle\big(\dot{P} + PA + A^\top P + C^\top PC + Q\big)X^{(u)}, X^{(u)}\big\rangle + 2\big\langle\big(B^\top P + D^\top PC + S\big)X^{(u)}, u\big\rangle \\
&\qquad + \big\langle\big(R + D^\top PD\big)u, u\big\rangle\Big] ds \\
&= \mathbb{E}\int_0^T \Big[\big\langle\Theta^\top\big(R + D^\top PD\big)\Theta X^{(u)}, X^{(u)}\big\rangle - 2\big\langle\big(R + D^\top PD\big)\Theta X^{(u)}, u\big\rangle + \big\langle\big(R + D^\top PD\big)u, u\big\rangle\Big] ds \\
&= \mathbb{E}\int_0^T \big\langle\big(R + D^\top PD\big)\big(u - \Theta X^{(u)}\big), u - \Theta X^{(u)}\big\rangle ds.
\end{aligned}
$$

Noting (4.32) and making use of Lemma 2.5, we obtain that

$$
J^0(0,0;u(\cdot)) = \mathbb{E}\int_0^T \big\langle\big(R + D^\top PD\big)\big(u - \Theta X^{(u)}\big), u - \Theta X^{(u)}\big\rangle ds \geqslant \lambda\gamma\mathbb{E}\int_0^T |u(s)|^2 ds, \quad \forall u(\cdot) \in \mathcal{U}[0,T],
$$

for some $\gamma > 0$. Then (i) holds. $\square$

Remark 4.7. From the first part of the proof of Theorem 4.6, we see that if (4.2) holds, then the strongly regular solution of (4.7) satisfies (4.11) with the same constant $\lambda > 0$.

Combining Theorem 4.4 and Theorem 4.6, we obtain the following corollary.

Corollary 4.8. *Let* (H1)–(H2) *and* (4.2) *hold. Then Problem* (SLQ) *is uniquely open-loop solvable at any* $(t,x) \in [0,T) \times \mathbb{R}^n$ *with the open-loop optimal control* $u^*(\cdot)$ *being of a state feedback form:*

$$
u^*(\cdot) = -(R + D^\top PD)^{-1}(B^\top P + D^\top PC + S)X^* - (R + D^\top PD)^{-1}(B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho), \tag{4.33}
$$

where $P(\cdot)$ *is the unique strongly regular solution of* (4.7) *with* $(\eta(\cdot), \zeta(\cdot))$ *being the adapted solution of* (4.15) *and* $X^*(\cdot)$ *being the solution of the following closed-loop system:*

$$
\begin{cases}
dX^*(s) = \Big\{\big[A - B(R + D^\top PD)^{-1}(B^\top P + D^\top PC + S)\big]X^* \\
\qquad\qquad - B(R + D^\top PD)^{-1}(B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho) + b\Big\}ds \\
\qquad\quad + \Big\{\big[C - D(R + D^\top PD)^{-1}(B^\top P + D^\top PC + S)\big]X^* \\
\qquad\qquad - D(R + D^\top PD)^{-1}(B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho) + \sigma\Big\}dW(s), \qquad s \in [t,T], \\
X^*(t) = x.
\end{cases} \tag{4.34}
$$

Proof. By Theorem 4.6, the Riccati equation (4.7) admits a unique strongly regular solution $P(\cdot) \in C([0,T];\mathbb{S}^n)$. Hence, the adapted solution $(\eta(\cdot), \zeta(\cdot))$ of (4.15) satisfies (4.16) automatically. Now applying Theorem 4.4 and noting the remark right after Definition 2.1, we get the desired result. $\square$

Remark 4.9. Under the assumptions of Corollary 4.8, when $b(\cdot), \sigma(\cdot), g, q(\cdot), \rho(\cdot) = 0$, the adapted solution of (4.15) is $(\eta(\cdot), \zeta(\cdot)) \equiv (0,0)$. Thus, for Problem $(\text{SLQ})^0$, the unique optimal control $u^*(\cdot)$ at initial pair $(t,x) \in [0,T) \times \mathbb{R}^n$ is given by

$$
u^*(\cdot) = -(R + D^\top PD)^{-1}(B^\top P + D^\top PC + S)X^*, \tag{4.35}
$$

with $P(\cdot)$ being the unique strongly regular solution of (4.7) and $X^*(\cdot)$ being the solution of the following closed-loop system:

$$
\begin{cases} dX^*(s) = \big[A - B(R + D^\top PD)^{-1}(B^\top P + D^\top PC + S)\big]X^* ds \\ \qquad\qquad + \big[C - D(R + D^\top PD)^{-1}(B^\top P + D^\top PC + S)\big]X^* dW(s), \qquad s \in [t, T], \\ X^*(t) = x. \end{cases} \tag{4.36}
$$

Moreover, by (4.18), the value function of Problem (SLQ)^0 is given by

$$
V^0(t, x) = \langle P(t)x, x\rangle, \qquad (t, x) \in [0, T] \times \mathbb{R}^n. \tag{4.37}
$$

5 Finiteness of Problem (SLQ) and Convexity of Cost Functional

We have seen that the uniform convexity of the cost functional implies the open-loop and closed-loop solvabilities of Problem (SLQ). We expect that the finiteness of Problem (SLQ) should be closely related to the convexity of the cost functional. A main purpose of this section is to make this clear. Other relevant issues will also be discussed. First, we introduce the following:

$$
\Lambda(s, P(\cdot)) = \begin{pmatrix} \dot P(s)+P(s)A(s)+A(s)^\top P(s)+C(s)^\top P(s)C(s)+Q(s) & P(s)B(s)+C(s)^\top P(s)D(s)+S(s)^\top \\ B(s)^\top P(s)+D(s)^\top P(s)C(s)+S(s) & R(s)+D(s)^\top P(s)D(s) \end{pmatrix}, \tag{5.1}
$$

for any $P(\cdot) \in AC(t, T; \mathbb{S}^n)$ which is the set of all absolutely continuous functions $P : [t, T] \to \mathbb{S}^n$. Let

$$
\mathcal{P}[t, T] = \Big\{P(\cdot) \in AC(t, T; \mathbb{S}^n) \;\big|\; P(T) \leqslant G,\ \Lambda(s, P(\cdot)) \geqslant 0,\ \text{a.e. } s \in [t, T]\Big\}.
$$

We have the following result.

Proposition 5.1. *Let* (H1)–(H2) *hold, and $t \in [0, T)$ be given. Among the following statements:*

(i) *Problem* (SLQ) *is finite at t.*

(ii) *Problem* (SLQ)^0 *is finite at t.*

(iii) *There exists a $P(t) \in \mathbb{S}^n$ such that*

$$
V^0(t, x) = \langle P(t)x, x\rangle, \qquad \forall x \in \mathbb{R}^n. \tag{5.2}
$$

(iv) *The map $u(\cdot) \mapsto J(t, x; u(\cdot))$ is convex, for any $x \in \mathbb{R}^n$.*

(v) $\mathcal{P}[t, T] \neq \emptyset$.

the following implications hold:

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv); (v) $\Rightarrow$ (ii).

Proof. (i) $\Rightarrow$ (ii). By Proposition 3.1, for any $x \in \mathbb{R}^n$ and $u(\cdot) \in \mathcal{U}[t, T]$, we have

$$
\begin{aligned} V(t, x) + V(t, -x) &\leqslant J(t, x; u(\cdot)) + J(t, -x; -u(\cdot)) \\ &= 2\Big[\big\langle M_2(t)u, u\big\rangle + 2\big\langle M_1(t)x, u\big\rangle + \big\langle M_0(t)x, x\big\rangle + c_t\Big] \\ &= 2\Big[J^0(t, x; u(\cdot)) + c_t\Big], \end{aligned}
$$

which implies (ii).

(ii) $\Rightarrow$ (iii) can be shown by a simple adoption of the well-known result in the deterministic case (see [11, 4]).

(iii) $\Rightarrow$ (iv). By Corollary 3.3, if $u(\cdot)\mapsto J(t,x;u(\cdot))$ is not convex, then $J^0(t,0;u(\cdot))<0$ for some $u(\cdot)\in\mathcal{U}[t,T]$. By Corollary 3.2, we have

$$J^0(t,x;\lambda u(\cdot))=J^0(t,x;0)+\lambda^2J^0(t,0;u(\cdot))+\lambda\mathbb{E}\int_t^T\big\langle\mathcal{D}J^0(t,x;0)(s),u(s)\big\rangle ds,\qquad\forall\lambda\in\mathbb{R}^n.$$

Letting $\lambda\to\infty$, we obtain

$$V^0(t,x)\leqslant\lim_{\lambda\to\infty}J^0(t,x;\lambda u(\cdot))=-\infty,$$

which is a contradiction.

(v) $\Rightarrow$ (ii). For any $(t,x)\in[0,T)\times\mathbb{R}^n$, $u(\cdot)\in\mathcal{U}[t,T]$, and any $P(\cdot)\in AC(t,T;\mathbb{S}^n)$, one has

$$\begin{aligned}&\mathbb{E}\big\langle P(T)X(T),X(T)\big\rangle-\big\langle P(t)x,x\big\rangle\\&=\mathbb{E}\int_t^T\Big\{\Big\langle\Big[\dot P(s)+P(s)A(s)+A(s)^\top P(s)+C(s)^\top P(s)C(s)\Big]X(s),X(s)\Big\rangle\\&\qquad\qquad+2\Big\langle\Big[B(s)^\top P(s)+D(s)^\top P(s)C(s)\Big]X(s),u(s)\Big\rangle+\big\langle D(s)^\top P(s)D(s)u(s),u(s)\big\rangle\Big\}ds.\end{aligned}$$

Hence, if $\mathcal{P}[t,T]\neq\emptyset$, then by taking $P(\cdot)\in\mathcal{P}[t,T]$, one has

$$\begin{aligned}J^0(t,x;u(\cdot))&=\mathbb{E}\left\{\big\langle GX(T),X(T)\big\rangle+\int_t^T\left\langle\begin{pmatrix}Q(s)&S(s)^\top\\S(s)&R(s)\end{pmatrix}\begin{pmatrix}X(s)\\u(s)\end{pmatrix},\begin{pmatrix}X(s)\\u(s)\end{pmatrix}\right\rangle ds\right\}\\&=\big\langle P(t)x,x\big\rangle+\mathbb{E}\left\{\big\langle\big[G-P(T)\big]X(T),X(T)\big\rangle+\int_t^T\left\langle\Lambda(s,P(\cdot))\begin{pmatrix}X(s)\\u(s)\end{pmatrix},\begin{pmatrix}X(s)\\u(s)\end{pmatrix}\right\rangle ds\right\}\\&\geqslant\big\langle P(t)x,x\big\rangle,\qquad\forall u(\cdot)\in\mathcal{U}[t,T].\end{aligned}$$

This implies that the corresponding Problem (SLQ)^0 is finite at t. $\square$

It is worth to point out that the convexity of the map $u(\cdot)\mapsto J^0(t,x;u(\cdot))$ is not sufficient for the finiteness of Problem (SLQ)^0. We present the following example (see also [21] for an example of a quadratic functional in Hilbert space).

Example 5.2. Consider the following one-dimensional controlled SDE:

$$\begin{cases}dX(s)=u(s)ds+X(s)dW(s),\quad s\in[t,1],\\X(t)=x,\end{cases}\tag{5.3}$$

and the cost functional:

$$J^0(t,x;u(\cdot))=\mathbb{E}\left[-X(1)^2+\int_t^1e^{1-s}u(s)^2ds\right].\tag{5.4}$$

We claim that

$$J^0(0,0;u(\cdot))\geqslant0,\qquad\forall u(\cdot)\in\mathcal{U}[0,T],\tag{5.5}$$

which, by Corollary 3.3, is equivalent to the convexity of $u(\cdot)\mapsto J^0(0,x;u(\cdot))$, but

$$V^0(0,x)=-\infty,\qquad\forall x\neq0.\tag{5.6}$$

To show the above, let $u(\cdot)\in\mathcal{U}[0,T]$ and $X(\cdot)\equiv X(\cdot\,;0,x,u(\cdot))$ be the solution of (5.3) with $t=0$. By the variation of constants formula,

$$X(s)=xe^{W(s)-\frac{1}{2}s}+e^{W(s)-\frac{1}{2}s}\int_0^s e^{\frac{1}{2}r-W(r)}u(r)dr,\qquad s\in[0,1].$$

Taking $x=0$ and noting that $e^{2[W(1)-W(r)]-(1-r)}$ is independent of $\mathcal{F}_r$, we have

$$\begin{aligned}\mathbb{E}\big[X(1)^2\big]&=\mathbb{E}\left[\int_0^1 e^{W(1)-W(r)-\frac{1}{2}(1-r)}u(r)dr\right]^2\leqslant\mathbb{E}\int_0^1 e^{2[W(1)-W(r)]-(1-r)}u(r)^2dr\\&=\int_0^1\mathbb{E}e^{2[W(1)-W(r)]-(1-r)}\mathbb{E}[u(r)^2]dr=\int_0^1 e^{1-r}\mathbb{E}[u(r)^2]dr=\mathbb{E}\int_0^1 e^{1-r}u(r)^2dr,\end{aligned}$$

and hence

$$J^0(0,0;u(\cdot))=\mathbb{E}\left[-X(1)^2+\int_0^1 e^{1-s}u(s)^2ds\right]\geqslant 0,\qquad\forall u(\cdot)\in\mathcal{U}[0,T].$$

On the other hand, taking $x\neq 0$ and $u(s)=\lambda e^{W(s)-\frac{1}{2}s},\ \lambda\in\mathbb{R}$, we have

$$X(1)=(x+\lambda)e^{W(1)-\frac{1}{2}}.$$

Therefore,

$$\begin{aligned}J^0(0,x;u(\cdot))&=\mathbb{E}\left[-X(1)^2+\int_0^1 e^{1-s}u(s)^2ds\right]=-\mathbb{E}\Big[(x+\lambda)^2e^{2W(1)-1}\Big]+\lambda^2\mathbb{E}\int_0^1 e^{1-s}e^{2W(s)-s}ds\\&=-(x+\lambda)^2e+\lambda^2e=-(x^2+2\lambda x)e.\end{aligned}$$

Letting $|\lambda|\to\infty$, with $\lambda x>0$, in the above, we obtain $V^0(0,x)=-\infty$. This proves our claim.

The above example tells us that, besides the convexity of $u(\cdot)\mapsto J^0(t,x;u(\cdot))$, one needs some additional condition(s) in order to get the finiteness of Problem (SLQ)^0 at t. To find such a condition, let us make some observations. Suppose $u(\cdot)\mapsto J^0(0,x;u(\cdot))$ is convex, which, by Corollary 3.3, is equivalent to the following:

$$J^0(0,0;u(\cdot))\geqslant 0,\qquad\forall u(\cdot)\in\mathcal{U}[0,T].\tag{5.7}$$

Then for any $\varepsilon>0$, consider state equation (1.1) (with $b(\cdot),\sigma(\cdot)=0$) and the following cost functional:

$$\begin{aligned}J^0_\varepsilon(t,x;u(\cdot))&\triangleq\mathbb{E}\left\{\langle GX(T),X(T)\rangle+\int_t^T\left\langle\begin{pmatrix}Q(s)&S(s)^\top\\S(s)&R(s)+\varepsilon I\end{pmatrix}\begin{pmatrix}X(s)\\u(s)\end{pmatrix},\begin{pmatrix}X(s)\\u(s)\end{pmatrix}\right\rangle ds\right\}\\&=J^0(t,x;u(\cdot))+\varepsilon\mathbb{E}\int_0^T|u(s)|^2ds.\end{aligned}\tag{5.8}$$

Denote the corresponding optimal control problem and value function by Problem (SLQ)$^0_\varepsilon$ and $V^0_\varepsilon(\cdot\,,\cdot)$, respectively. By Corollary 3.3 and the convexity of $u(\cdot)\mapsto J^0(0,0;u(\cdot))$, one has

$$J^0_\varepsilon(0,0;u(\cdot))=J^0(0,0;u(\cdot))+\varepsilon\mathbb{E}\int_0^T|u(s)|^2ds\geqslant\varepsilon\mathbb{E}\int_0^T|u(s)|^2ds,\qquad\forall u(\cdot)\in\mathcal{U}[0,T],$$

i.e., $u(\cdot)\mapsto J^0_\varepsilon(0,0;u(\cdot))$ is uniformly convex. Hence, it follows from Theorem 4.6 that the following Riccati equation:

$$\begin{cases}\dot P_\varepsilon+P_\varepsilon A+A^\top P_\varepsilon+C^\top P_\varepsilon C+Q\\\qquad-(P_\varepsilon B+C^\top P_\varepsilon D+S^\top)(R+\varepsilon I+D^\top P_\varepsilon D)^{-1}(B^\top P_\varepsilon+D^\top P_\varepsilon C+S)=0,\qquad\text{a.e. }\ s\in[0,T],\\P_\varepsilon(T)=G,\end{cases}\tag{5.9}$$

admits a unique strongly regular solution $P_\varepsilon(\cdot) \in C([0,T];\mathbb{S}^n)$ such that (noting Remark 4.7)

$$R(t) + \varepsilon I + D(t)^\top P_\varepsilon(t) D(t) \geqslant \varepsilon I, \qquad \text{a.e. } t \in [0,T]. \tag{5.10}$$

Now, we are ready to state and prove the following result which is a characterization of the finiteness of Problem (SLQ)^0.

Theorem 5.3. *Let* (H1)–(H2) *and* (5.7) *hold. For any* $\varepsilon > 0$, *let* $P_\varepsilon(\cdot)$ *be the unique strongly regular solution of the Riccati equation* (5.9). *Then Problem* (SLQ)^0 *is finite if and only if* $\{P_\varepsilon(0)\}_{\varepsilon>0}$ *is bounded from below. In this case, the limit*

$$\lim_{\varepsilon\to 0} P_\varepsilon(t) = P(t), \qquad \forall t \in [0,T], \tag{5.11}$$

exists, and (5.2) *holds. Moreover,*

$$R(t) + D(t)^\top P(t) D(t) \geqslant 0, \qquad \text{a.e. } \ t \in [0,T], \tag{5.12}$$

and

$$N(t) \leqslant P(t) \leqslant M_0(t), \qquad \forall t \in [0,T], \tag{5.13}$$

where $M_0(\cdot)$ *is the solution to the Lyapunov equation* (3.8), and

$$N(t) = \big[\Phi_A(t)^\top\big]^{-1} \left\{ P(0) - \int_0^t \Phi_A(s)^\top \Big[C(s)^\top M_0(s) C(s) + Q(s)\Big] \Phi_A(s) ds \right\} \Phi_A(t)^{-1}, \tag{5.14}$$

with $\Phi_A(\cdot)$ being the solution to the following:

$$\begin{cases} \dot{\Phi}_A(s) = A(s)\Phi_A(s), \qquad s \geqslant 0, \\ \Phi_A(0) = I. \end{cases} \tag{5.15}$$

In particular, if Problem (SLQ)^0 is finite at $t = 0$, then it is finite.

Proof. Necessity. Suppose Problem (SLQ)^0 is finite and let $P : [0,T] \to \mathbb{S}^n$ such that (5.2) holds. For any $\varepsilon_2 > \varepsilon_1 > 0$, we have

$$J^0_{\varepsilon_2}(t,x;u(\cdot)) \geqslant J^0_{\varepsilon_1}(t,x;u(\cdot)) \geqslant J^0(t,x;u(\cdot)), \qquad \forall (t,x) \in [0,T]\times\mathbb{R}^n, \quad \forall u(\cdot) \in \mathcal{U}[t,T].$$

Hence (noting Remark 4.9),

$$\begin{aligned} \langle P_{\varepsilon_2}(t)x, x\rangle = V^0_{\varepsilon_2}(t,x) &= \inf_{u(\cdot)\in\mathcal{U}[t,T]} J^0_{\varepsilon_2}(t,x;u(\cdot)) \\ &\geqslant \inf_{u(\cdot)\in\mathcal{U}[t,T]} J^0_{\varepsilon_1}(t,x;u(\cdot)) = V^0_{\varepsilon_1}(t,x) = \langle P_{\varepsilon_1}(t)x,x\rangle \\ &\geqslant \inf_{u(\cdot)\in\mathcal{U}[t,T]} J^0(t,x;u(\cdot)) = V^0(t,x) = \langle P(t)x,x\rangle, \qquad \forall (t,x)\in[0,T]\times\mathbb{R}^n. \end{aligned}$$

Thus $\{P_\varepsilon(t)\}_{\varepsilon>0}$ is a nondecreasing sequence with lower bound $P(t)$ and therefore has a limit $\bar{P}(t)$ with

$$\bar{P}(t) \equiv \lim_{\varepsilon\to0} P_\varepsilon(t) \geqslant P(t), \qquad \forall t \in [0,T]. \tag{5.16}$$

On the other hand, for any $\delta > 0$, we can find a $u^\delta(\cdot) \in \mathcal{U}[t,T]$, such that

$$V^0_\varepsilon(t,x) \leqslant J^0(t,x;u^\delta(\cdot)) + \varepsilon\mathbb{E}\int_t^T |u^\delta(s)|^2 ds \leqslant V^0(t,x) + \delta + \varepsilon\mathbb{E}\int_t^T |u^\delta(s)|^2 ds.$$

Letting $\varepsilon \to 0$, we obtain that

$$\langle \bar{P}(t)x, x\rangle = \lim_{\varepsilon\to 0}\langle P_\varepsilon(t)x, x\rangle = \lim_{\varepsilon\to 0} V_\varepsilon^0(t,x) \leqslant V^0(t,x) + \delta = \langle P(t)x, x\rangle + \delta,$$

from which we see that

$$\langle \bar{P}(t)x, x\rangle \leqslant \langle P(t)x, x\rangle \qquad \forall (t,x) \in [0,T] \times \mathbb{R}^n. \tag{5.17}$$

Combining (5.16)–(5.17), we obtain (5.11) with $P(\cdot)$ satisfying (5.2). Moreover, letting $\varepsilon \to 0$ in (5.10), we obtain (5.12).

Sufficiency. Suppose there exists a $\beta \in \mathbb{R}$ such that

$$P_\varepsilon(0) \geqslant \beta I, \qquad \forall \varepsilon > 0,$$

then for any $x \in \mathbb{R}^n$ and $u(\cdot) \in \mathcal{U}[0,T]$, we have

$$J^0(0,x;u(\cdot)) + \varepsilon\mathbb{E}\int_0^T |u(s)|^2 ds \geqslant V_\varepsilon^0(0,x) = \langle P_\varepsilon(0)x, x\rangle \geqslant \beta|x|^2, \qquad \forall \varepsilon > 0.$$

Letting $\varepsilon \to 0$ in the above, we obtain

$$J^0(0,x;u(\cdot)) \geqslant \beta|x|^2, \qquad \forall x \in \mathbb{R}^n, \quad \forall u(\cdot) \in \mathcal{U}[0,T],$$

which implies the finiteness of Problem (SLQ)^0 at $t = 0$.

Now, let $P(0) \in \mathbb{S}^n$ such that $V^0(0,x) = \langle P(0)x, x\rangle, \forall x \in \mathbb{R}^n$. Then

$$P(0) \leqslant P_\varepsilon(0), \qquad \forall \varepsilon > 0. \tag{5.18}$$

Also, by Remark 4.9 and Proposition 3.1,

$$\langle P_\varepsilon(t)x, x\rangle = V_\varepsilon^0(t,x) \leqslant J_\varepsilon^0(t,x;0) = J^0(t,x;0) = \langle M_0(t)x, x\rangle, \qquad \forall x \in \mathbb{R}^n.$$

This leads to

$$P_\varepsilon(t) \leqslant M_0(t), \qquad t \in [0,T], \qquad \forall \varepsilon > 0. \tag{5.19}$$

On the other hand, let $\Phi_A(\cdot)$ be the solution of (5.15) and set

$$\Pi_\varepsilon \triangleq (P_\varepsilon B + C^\top P_\varepsilon D + S^\top)(R_\varepsilon + D^\top P_\varepsilon D)^{-1}(B^\top P_\varepsilon + D^\top P_\varepsilon C + S) \geqslant 0.$$

Differentiating $s \mapsto \Phi_A(s)^\top P_\varepsilon(s)\Phi_A(s)$, we obtain

$$\frac{d}{ds}\Big[\Phi_A(s)^\top P_\varepsilon(s)\Phi_A(s)\Big] = \Phi_A(s)^\top\Big[\Pi_\varepsilon(s) - C(s)^\top P_\varepsilon(s)C(s) - Q(s)\Big]\Phi_A(s).$$

Hence, combining (5.18)–(5.19), we have

$$\begin{aligned}\Phi_A(t)^\top P_\varepsilon(t)\Phi_A(t) &= P_\varepsilon(0) + \int_0^t \Phi_A(s)^\top\Big[\Pi_\varepsilon(s) - C(s)^\top P_\varepsilon(s)C(s) - Q(s)\Big]\Phi_A(s)ds\\ &\geqslant P(0) - \int_0^t \Phi_A(s)^\top\Big[C(s)^\top M_0(s)C(s) + Q(s)\Big]\Phi_A(s)ds.\end{aligned}$$

Thus,

$$\begin{aligned}P_\varepsilon(t) &\geqslant \big[\Phi_A(t)^\top\big]^{-1}\left\{P(0) - \int_0^t \Phi_A(s)^\top\Big[C(s)^\top M_0(s)C(s) + Q(s)\Big]\Phi_A(s)ds\right\}\Phi_A(t)^{-1}\\ &\equiv N(t), \qquad t \in [0,T].\end{aligned} \tag{5.20}$$

Then using the same argument as in the previous paragraph, we can show that

$$J^0(t,x;u(\cdot)) \geqslant \langle N(t)x,x\rangle, \qquad \forall (t,x)\in[0,T]\times\mathbb{R}^n, \quad \forall u(\cdot)\in\mathcal{U}[t,T], \tag{5.21}$$

which implies the finiteness of Problem (SLQ)^0. Moreover, let $P:[0,T]\to\mathbb{S}^n$ such that (5.2) holds, then

$$\langle N(t)x,x\rangle \leqslant \inf_{u(\cdot)\in\mathcal{U}[t,T]} J^0(t,x;u(\cdot)) = \langle P(t)x,x\rangle \leqslant J^0(t,x;0) = \langle M_0(t)x,x\rangle, \qquad \forall (t,x)\in[0,T]\times\mathbb{R}^n,$$

and (5.13) follows.

Finally, if Problem (SLQ)^0 is finite at $t=0$, then (5.18) holds and the finiteness of Problem (SLQ)^0 therefore follows. $\square$

The following is another sufficient condition for the finiteness of Problem (SLQ)^0, which is a corollary of Theorem 5.3 and Proposition 5.1, (v) $\Rightarrow$ (ii).

Corollary 5.4. *Let* (H1)–(H2) *hold. If there exists a* $\Delta(\cdot)\in L^1(0,T;\mathbb{S}^n_+)$ *such that*

$$R + D^\top PD \geqslant \big(B^\top P + D^\top PC + S\big)\Delta^{-1}\big(PB + C^\top PD + S^\top\big), \qquad \text{a.e. } \ s\in[0,T], \tag{5.22}$$

where $P(\cdot)$ *is the solution of the following Lyapunov equation:*

$$\begin{cases} \dot P(s) + P(s)A(s) + A(s)^\top P(s) + C(s)^\top P(s)C(s) + Q(s) = \Delta(s), \qquad \text{a.e. } \ s\in[0,T], \\ P(T)\leqslant G. \end{cases} \tag{5.23}$$

Then Problem (SLQ)^0 *is finite.*

Proof. Under our condition, by Lemma 2.6, one has

$$\Lambda(s,P(\cdot)) = \begin{pmatrix} \Delta(s) & P(s)B(s)+C(s)^\top P(s)D(s)+S(s)^\top \\ B(s)^\top P(s)+D(s)^\top P(s)C(s)+S(s) & R(s)+D(s)^\top P(s)D(s) \end{pmatrix} \geqslant 0, \qquad \text{a.e. } \ s\in[0,T].$$

Hence, $P(\cdot)\in\mathcal{P}[0,T]$ and Problem (SLQ)^0 is finite at $t=0$. Then the finiteness of Problem (SLQ)^0 follows from Theorem 5.3 immediately. $\square$

We now return to the study of convexity of the map $u(\cdot)\mapsto J^0(t,0;u(\cdot))$. First, from the definition of $M_2(t)$ (see (3.13)), we see that $M_2(t)\geqslant 0$ if and only if

$$R(\cdot) \geqslant -\Big(\widehat L_t^* G \widehat L_t + L_t^* Q L_t + SL_t + L_t^* S^\top\Big), \tag{5.24}$$

with the right hand side being non-positive. Thus, unlike the well-known situation for the deterministic LQ problems (for which $R(\cdot)\geqslant 0$ is necessary for $M_2(t)\geqslant 0$ ([30])), $R(\cdot)$ does not have to be positive semi-definite. Actually, as shown by examples in [7, 30], $R(\cdot)$ could even be negative definite in some extent. Let us now take a closer look at this issue.

Note that when $u(\cdot)\mapsto J^0(0,0;u(\cdot))$ is convex, for any $\varepsilon>0$, the unique strongly regular solution $P_\varepsilon(\cdot)$ to the Riccati equation (5.9) satisfies (5.10) and (5.19). Hence,

$$R(t) + D(t)^\top M_0(t)D(t) \geqslant 0, \qquad \text{a.e. } t\in[0,T], \tag{5.25}$$

or equivalently,

$$\begin{aligned} R(t) + D(t)^\top\mathbb{E}\Big\{&\big[\Phi(T)\Phi(t)^{-1}\big]^\top G\big[\Phi(T)\Phi(t)^{-1}\big] \\ &+\int_t^T \big[\Phi(s)\Phi(t)^{-1}\big]^\top Q(s)\big[\Phi(s)\Phi(t)^{-1}\big]ds\Big\}D(t) \geqslant 0, \qquad \text{a.e. } t\in[0,T]. \end{aligned} \tag{5.26}$$

This is another necessary condition for the finiteness of Problem $(\mathrm{SLQ})^0$, which is easier to check. From (5.26), we see that if $R(\cdot)$ happens to be negative definite, then in order $u(\cdot)\mapsto J^0(0,0;u(\cdot))$ to be convex, it is necessary that $D(\cdot)$ is injective, and either G or $Q(\cdot)$ (or both) has to be positive enough to compensate. Note that $D(s)$ was assumed to be invertible in [23]. Therefore, in some sense, our result justifies the assumption of [23].

The following gives a little improvement when more restrictive conditions are assumed.

Proposition 5.5. *Let* (H1)–(H2) *hold. Suppose that*

$$B(\cdot)=0,\qquad C(\cdot)=0,\qquad S(\cdot)=0. \tag{5.27}$$

Then the map $u(\cdot)\mapsto J^0(0,0;u(\cdot))$ *is convex if and only if* (5.26) *holds. In this case, Problem* $(\mathrm{SLQ})^0$ *is closed-loop solvable.*

Proof. It suffices to prove the sufficiency. Note that in the current case, the corresponding Riccati equation becomes

$$\begin{cases}\dot P(s)+P(s)A(s)+A(s)^\top P(s)+Q(s)=0, \qquad \text{a.e. } s\in[0,T],\\ P(T)=G,\end{cases} \tag{5.28}$$

whose solution is $M_0(\cdot)$. If (5.26) holds, then it is easy to verify that $M_0(\cdot)$ is regular. Consequently, by Theorem 4.4, Problem $(\mathrm{SLQ})^0$ is closed-loop solvable, and hence $u(\cdot)\mapsto J^0(0,0;u(\cdot))$ is convex. $\square$

Note that in the case (5.27), we have

$$M_0(t)=\big[\Phi_A(T)\Phi_A(t)^{-1}\big]^\top G\big[\Phi_A(T)\Phi_A(t)^{-1}\big]+\int_t^T\big[\Phi_A(s)\Phi_A(t)^{-1}\big]^\top Q(s)\big[\Phi_A(s)\Phi_A(t)^{-1}\big]ds,$$

with $\Phi_A(\cdot)$ being the solution of (5.15). Hence, (5.26) can also be written as

$$\begin{aligned}R(t)+D(t)^\top\bigg\{&\big[\Phi_A(T)\Phi_A(t)^{-1}\big]^\top G\big[\Phi_A(T)\Phi_A(t)^{-1}\big]\\ &+\int_t^T\big[\Phi_A(s)\Phi_A(t)^{-1}\big]^\top Q(s)\big[\Phi_A(s)\Phi_A(t)^{-1}\big]ds\bigg\}D(t)\geqslant 0, \qquad \text{a.e. } t\in[0,T].\end{aligned} \tag{5.29}$$

As we pointed out earlier, Problem $(\mathrm{SLQ})^0$ may still be infinite at some initial pair (t,x) even if the cost functional is convex. In this case, by Theorem 5.3, the sequence $\{P_\varepsilon(t)\}_{\varepsilon>0}$ diverges. The following result is concerned with the divergence speed of $\{P_\varepsilon(t)\}_{\varepsilon>0}$.

Proposition 5.6. *Let* (H1)–(H2) *and* (5.7) *hold. For any* $\varepsilon>0$, *let* $P_\varepsilon(\cdot)$ *be the unique strongly regular solution of* (5.9). *Then for any* $\alpha\geqslant 1$ *and* $t\in[0,T]$, *the sequence* $\{\varepsilon^\alpha P_\varepsilon(t)\}_{\varepsilon>0}$ *converges.*

Proof. Let $\varepsilon>0$ and consider Problem $(\mathrm{SLQ})^0_\varepsilon$. By Proposition 3.1 (noting $M_2(t)\geqslant 0$),

$$\begin{aligned}J^0_\varepsilon(t,x;u(\cdot))&=J^0(t,x;u(\cdot))+\varepsilon\mathbb{E}\int_t^T|u(s)|^2ds\\ &=\big\langle[M_2(t)+\varepsilon I]u,u\big\rangle+2\big\langle M_1(t)x,u\big\rangle+\big\langle M_0(t)x,x\big\rangle\\ &=\Big|[M_2(t)+\varepsilon I]^{\frac12}u+[M_2(t)+\varepsilon I]^{-\frac12}M_1(t)x\Big|^2\\ &\quad+\big\langle M_0(t)x,x\big\rangle-\big\langle[M_2(t)+\varepsilon I]^{-1}M_1(t)x,M_1(t)x\big\rangle.\end{aligned}$$

Thus

$$\langle P_\varepsilon(t)x,x\rangle=\inf_{u(\cdot)\in\mathcal{U}[t,T]}J^0_\varepsilon(t,x;u(\cdot))=\big\langle M_0(t)x,x\big\rangle-\big\langle[M_2(t)+\varepsilon I]^{-1}M_1(t)x,M_1(t)x\big\rangle. \tag{5.30}$$

For any $\alpha\geqslant 1$, we have

$$0\leqslant\varepsilon^\alpha\big\langle[M_2(t)+\varepsilon I]^{-1}M_1(t)x,M_1(t)x\big\rangle\leqslant\varepsilon^{\alpha-1}\big\langle M_1(t)x,M_1(t)x\big\rangle\leqslant\big\langle M_1(t)x,M_1(t)x\big\rangle,\qquad\forall 0<\varepsilon\leqslant 1.$$

Consequently, for any $x\in\mathbb{R}^n$, $\{\varepsilon^\alpha\langle P_\varepsilon(t)x,x\rangle\}_{\varepsilon>0}$ has a convergent subsequence, and hence the sequence $\{\varepsilon^\alpha\langle P_\varepsilon(t)x,x\rangle\}_{\varepsilon>0}$ itself converges as $\varepsilon\to 0$ since it is nondecreasing. The result therefore follows. □

From Remark 4.7 and 4.9, we see that if the uniformly convex condition (4.2) holds, then Problem $(\text{SLQ})^0$ is finite and

$$R(s)+D(s)^\top P(s)D(s)\geqslant\lambda I,\qquad\text{a.e. }\; s\in[0,T], \tag{5.31}$$

where $P:[0,T]\to\mathbb{S}^n$ is the function such that (5.2) holds. The following result shows that the converse is also true.

Theorem 5.7. *Let* (H1)–(H2) *hold. Suppose Problem* $(\text{SLQ})^0$ *is finite and let* $P:[0,T]\to\mathbb{S}^n$ *such that* (5.2) *holds. If* (5.31) *holds for some* $\lambda>0$*, then* $P(\cdot)$ *solves the Riccati equation* (4.7)*. Consequently, the map* $u(\cdot)\mapsto J^0(0,0;u(\cdot))$ *is uniformly convex.*

Proof. For any $\varepsilon>0$, let $P_\varepsilon(\cdot)$ be the unique strongly regular solution of (5.9). By Theorem 5.3,

$$P_\varepsilon(t)\searrow P(t),\qquad\text{as}\quad\varepsilon\searrow 0,\quad\forall t\in[0,T].$$

Note that $P_\varepsilon(\cdot)\leqslant M_0(\cdot)$ $\forall\varepsilon>0$ and by (5.13), $P(\cdot)$ is bounded. Thus, $\{P_\varepsilon(t)\}_{\varepsilon>0}$ is uniformly bounded. Also, we have

$$R(s)+D(s)^\top P_\varepsilon(s)D(s)\geqslant R(s)+D(s)^\top P(s)D(s)\geqslant\lambda I,\qquad\text{a.e. }\; s\in[0,T],\quad\forall\varepsilon>0.$$

Then it follows from the dominated convergence theorem that

$$P_\varepsilon A+A^\top P_\varepsilon+C^\top P_\varepsilon C+Q-(P_\varepsilon B+C^\top P_\varepsilon D+S^\top)(R+\varepsilon I+D^\top P_\varepsilon D)^{-1}(B^\top P_\varepsilon+D^\top P_\varepsilon C+S)\equiv\Lambda_\varepsilon$$

converges to

$$PA+A^\top P+C^\top PC+Q-(PB+C^\top PD+S^\top)(R+D^\top PD)^{-1}(B^\top P+D^\top PC+S)\equiv\Lambda$$

in L^1 as $\varepsilon\to 0$. Therefore,

$$P(t)=\lim_{\varepsilon\to 0}P_\varepsilon(t)=G+\lim_{\varepsilon\to 0}\int_t^T\Lambda_\varepsilon(s)ds=G+\int_t^T\Lambda(s)ds,$$

which, together with (5.31), implies that $P(\cdot)$ is a strongly regular solution of (4.7). Consequently, by Theorem 4.6, $u(\cdot)\mapsto J^0(0,0;u(\cdot))$ is uniformly convex. □

We now look at the following case:

$$D(\cdot)=0,\qquad R(\cdot)\gg 0. \tag{5.32}$$

Note that although $D(\cdot)=0$, since $C(\cdot)$ is not necessarily zero, our state equation is still an SDE. For such a case, the above results can be restated as follows.

Theorem 5.8. *Let* (H1)–(H2) *and* (5.32) *hold. Then the following statements are equivalent:*

(i) *Problem* (SLQ) *is finite at* $t = 0$*;*

(ii) *Problem* $(\text{SLQ})^0$ *is finite at* $t = 0$*;*

(iii) *The map* $u(\cdot) \mapsto J^0(0, 0; u(\cdot))$ *is uniformly convex;*

(iv) *The following Riccati equation*

$$
\begin{cases} \dot{P} + PA + A^\top P + C^\top PC + Q - (PB + S^\top)R^{-1}(B^\top P + S) = 0, \qquad \text{a.e.} \quad s \in [0, T], \\ P(T) = G, \end{cases} \tag{5.33}
$$

admits a unique solution $P(\cdot) \in C([0, T]; \mathbb{S}^n)$*;*

(v) *Problem* (SLQ) *is uniquely closed-loop solvable;*

(vi) *Problem* (SLQ) *is uniquely open-loop solvable.*

Proof. (i) $\Rightarrow$ (ii) follows from Proposition 5.1.

(ii) $\Rightarrow$ (iii). By Theorem 5.3, Problem $(\text{SLQ})^0$ is finite. Since $D(\cdot) = 0, R(\cdot) \gg 0$, (5.31) holds for some $\lambda > 0$, and the result follows from Theorem 5.7.

(iii) $\Leftrightarrow$ (iv). In the case of (5.32), the corresponding Riccati equation becomes (5.33). If $P(\cdot) \in C([0, T]; \mathbb{S}^n)$ is a solution of (5.33), then it is automatically strongly regular. Thus, by Theorem 4.6, we obtain the equivalence of (iii) and (iv).

(iii) $\Rightarrow$ (vi) follows from Proposition 4.1 and (iv) $\Rightarrow$ (v) follows from Theorem 4.4.

Finally, (v) $\Rightarrow$ (i) and (vi) $\Rightarrow$ (i) are trivial. $\square$

An interesting point of the above is that under condition (5.32), finiteness of Problem (SLQ) implies the closed-loop solvability of Problem (SLQ). In the deterministic case, such a fact was firstly revealed in [31] for two-person zero-sum differential games, and was proved in [29] for deterministic LQ problems by means of Fredholm operators.

6 Minimizing Sequences and Open-Loop Solvabilities

In Section 4, we showed that under the uniform convexity condition (4.2), Problem (SLQ) is open-loop solvable and the open-loop optimal control has a linear state feedback representation. In this section, we study the open-loop solvability of Problem (SLQ) without the the uniform convexity condition.

First we construct a minimizing sequence for Problem (SLQ) when it is finite.

Theorem 6.1. *Let* (H1)–(H2) *hold. Suppose Problem* (SLQ) *is finite. For any* $\varepsilon > 0$, *let* $P_\varepsilon(\cdot)$ *be the unique strongly regular solution to the Riccati equation* (5.9)*. Further, let* $(\eta_\varepsilon(\cdot), \zeta_\varepsilon(\cdot))$ *and* $X_\varepsilon(\cdot) \equiv X_\varepsilon(\cdot\,; t, x)$ *be the (adapted) solutions to the following BSDE and closed-loop system, respectively:*

$$
\begin{cases} d\eta_\varepsilon(s) = -\Big[(A + B\Theta_\varepsilon)^\top \eta_\varepsilon + (C + D\Theta_\varepsilon)^\top \zeta_\varepsilon + (C + D\Theta_\varepsilon)^\top P_\varepsilon \sigma - \Theta_\varepsilon^\top \rho + P_\varepsilon b + q\Big] ds \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \zeta_\varepsilon dW(s), \qquad s \in [0, T], \\ \eta_\varepsilon(T) = g, \end{cases} \tag{6.1}
$$

$$
(6.2) \qquad \begin{cases} dX_\varepsilon(s) = \Big[(A+B\Theta_\varepsilon)X_\varepsilon + Bv_\varepsilon + b\Big]ds + \Big[(C+D\Theta_\varepsilon)X_\varepsilon + Dv_\varepsilon + \sigma\Big]dW(s), \qquad s \in [t,T], \\ X_\varepsilon(t) = x, \end{cases}
$$

where

$$
(6.3) \qquad \begin{cases} \Theta_\varepsilon = -\big(R + \varepsilon I + D^\top P_\varepsilon D\big)^{-1}\big(B^\top P_\varepsilon + D^\top P_\varepsilon C + S\big), \\ v_\varepsilon = -\big(R + \varepsilon I + D^\top P_\varepsilon D\big)^{-1}\big(B^\top \eta_\varepsilon + D^\top \zeta_\varepsilon + D^\top P_\varepsilon \sigma + \rho\big). \end{cases}
$$

Then

$$
(6.4) \qquad u_\varepsilon(\cdot) \triangleq \Theta_\varepsilon(\cdot)X_\varepsilon(\cdot) + v_\varepsilon(\cdot), \qquad \varepsilon > 0
$$

is a minimizing sequence of $u(\cdot) \mapsto J(t,x;u(\cdot))$*:*

$$
(6.5) \qquad \lim_{\varepsilon\to 0} J(t,x;u_\varepsilon(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[t,T]} J(t,x;u(\cdot)) = V(t,x).
$$

Proof. For any $\varepsilon > 0$, consider state equation (1.1) and the following cost functional:

$$
(6.6) \qquad J_\varepsilon(t,x;u(\cdot)) \triangleq J(t,x;u(\cdot)) + \varepsilon\mathbb{E}\int_t^T |u(s)|^2 ds.
$$

Denote the above problem by Problem $(\text{SLQ})_\varepsilon$ and the corresponding value function by $V_\varepsilon(\cdot\,,\cdot)$. By Corollary 4.8, $u_\varepsilon(\cdot)$ defined by (6.4) is the unique optimal control of Problem $(\text{SLQ})_\varepsilon$ at $(t,x) \in [0,T)\times\mathbb{R}^n$. Note that

$$
\begin{aligned} \varepsilon\mathbb{E}\int_t^T |u_\varepsilon(s)|^2 ds &= J_\varepsilon(t,x;u_\varepsilon(\cdot)) - J(t,x;u_\varepsilon(\cdot)) = V_\varepsilon(t,x) - J(t,x;u_\varepsilon(\cdot)) \\ &\leqslant V_\varepsilon(t,x) - V(t,x) \to 0 \qquad \text{as} \quad \varepsilon \to 0. \end{aligned}
$$

Thus,

$$
\lim_{\varepsilon\to 0} J(t,x;u_\varepsilon(\cdot)) = \lim_{\varepsilon\to 0}\left[V_\varepsilon(t,x) - \varepsilon\mathbb{E}\int_t^T |u_\varepsilon(s)|^2 ds\right] = V(t,x).
$$

The proof is completed. □

Using the minimizing sequence constructed in Theorem 6.1, the open-loop solvability of Problem (SLQ) can be characterized as follows.

Theorem 6.2. *Let* (H1)–(H2) *hold. Suppose* $u(\cdot) \mapsto J^0(0,0;u(\cdot))$ *is convex. Let* $(t,x) \in [0,T)\times\mathbb{R}^n$ *and* $\{u_\varepsilon(\cdot)\}_{\varepsilon>0}$ *be the sequence defined by* (6.4)*. Then the following statements are equivalent:*

(i) *Problem* (SLQ) *is open-loop solvable at* (t,x)*;*

(ii) *The sequence* $\{u_\varepsilon(\cdot)\}_{\varepsilon>0}$ *admits a weakly convergent subsequence;*

(iii) *The sequence* $\{u_\varepsilon(\cdot)\}_{\varepsilon>0}$ *admits a strongly convergent subsequence.*

In this case, the weak (strong) limit of any weakly (strongly) convergent subsequence of $\{u_\varepsilon(\cdot)\}_{\varepsilon>0}$ *is an open-loop optimal control of Problem* (SLQ) *at* (t,x)*.*

To prove Theorem 6.2, we need the following lemma.

Lemma 6.3. *Let* $\mathcal{H}$ *be a Hilbert space with norm* $\|\cdot\|$ *and* $\theta,\theta_n \in \mathcal{H}$, $n = 1,2,\cdots$.

(i) *If* $\theta_n \to \theta$ *weakly, then* $\|\theta\| \leqslant \varliminf_{n\to\infty} \|\theta_n\|$.

(ii) $\theta_n \to \theta$ *strongly if and only if*

$$\|\theta_n\| \to \|\theta\| \qquad \textit{and} \qquad \theta_n \to \theta \quad \textit{weakly}.$$

Proof. (i) By the Hahn-Banach theorem, we can choose a $\theta^* \in \mathcal{H}$ with $\|\theta^*\| = 1$ such that $\langle \theta^*, \theta \rangle = \|\theta\|$. Thus,

$$\|\theta\| = \langle \theta^*, \theta \rangle = \lim_{n\to\infty} \langle \theta^*, \theta_n \rangle \leqslant \varliminf_{n\to\infty} \|\theta_n\|.$$

(ii) The necessity is obvious. Now suppose $\|\theta_n\| \to \|\theta\|$ and $\theta_n \to \theta$ weakly. Then

$$\|\theta_n - \theta\| = \|\theta_n\|^2 - 2\big\langle \theta, \theta_n \big\rangle + \|\theta\|^2 \to 0 \qquad \text{as} \quad n \to \infty.$$

This completes the proof. $\square$

Proof of Theorem 6.2. (i) $\Rightarrow$ (ii), and (i) $\Rightarrow$ (iii). Let $v^*(\cdot)$ be an open-loop optimal control of Problem (SLQ) at (t,x). By Corollary 4.8, for any $\varepsilon > 0$, $u_\varepsilon(\cdot)$ defined by (6.4) is the unique optimal control of Problem (SLQ)_ε at (t,x) and

$$V_\varepsilon(t,x) = J_\varepsilon(t,x;u_\varepsilon(\cdot)) \geqslant V(t,x) + \varepsilon \mathbb{E}\int_t^T |u_\varepsilon(s)|^2 ds. \tag{6.7}$$

Also, we have

$$V_\varepsilon(t,x) \leqslant J_\varepsilon(t,x;v^*(\cdot)) = V(t,x) + \varepsilon \mathbb{E}\int_t^T |v^*(s)|^2 ds. \tag{6.8}$$

Combining (6.7)–(6.8), we have

$$\mathbb{E}\int_t^T |u_\varepsilon(s)|^2 ds \leqslant \frac{V_\varepsilon(t,x) - V(t,x)}{\varepsilon} \leqslant \mathbb{E}\int_t^T |v^*(s)|^2 ds, \qquad \forall \varepsilon > 0. \tag{6.9}$$

Thus, $\{u_\varepsilon(\cdot)\}_{\varepsilon>0}$ is bounded in the Hilbert space $\mathcal{U}[t,T] = L^2_{\mathbb{F}}(t,T;\mathbb{R}^m)$ and hence admits a weakly convergent subsequence $\{u_{\varepsilon_k}(\cdot)\}_{k\geqslant 1}$. Let $u^*(\cdot)$ be the weak limit of $\{u_{\varepsilon_k}(\cdot)\}_{k\geqslant 1}$. Since $u(\cdot) \mapsto J(t,x;u(\cdot))$ is convex and continuous, it is hence sequentially weakly lower semi-continuous. Thus (noting (6.5)),

$$V(t,x) \leqslant J(t,x;u^*(\cdot)) \leqslant \varliminf_{k\to\infty} J(t,x;u_{\varepsilon_k}(\cdot)) = V(t,x),$$

which implies that $u^*(\cdot)$ is also an open-loop optimal control of Problem (SLQ) at (t,x). Now replacing $v^*(\cdot)$ with $u^*(\cdot)$ in (6.9), we have

$$\mathbb{E}\int_t^T |u_\varepsilon(s)|^2 ds \leqslant \mathbb{E}\int_t^T |u^*(s)|^2 ds, \qquad \forall \varepsilon > 0. \tag{6.10}$$

Also, by Lemma 6.3, part (i),

$$\mathbb{E}\int_t^T |u^*(s)|^2 ds \leqslant \varliminf_{k\to\infty} \mathbb{E}\int_t^T |u_{\varepsilon_k}(s)|^2 ds. \tag{6.11}$$

Combining (6.10)–(6.11), we see that

$$\mathbb{E}\int_t^T |u^*(s)|^2 ds = \lim_{k\to\infty} \mathbb{E}\int_t^T |u_{\varepsilon_k}(s)|^2 ds.$$

Then it follows from Lemma 6.3, part (ii), that $\{u_{\varepsilon_k}(\cdot)\}_{k\geqslant 1}$ converges to $u^*(\cdot)$ strongly.

(iii) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i). Let $\{u_{\varepsilon_k}(\cdot)\}_{k\geqslant 1}$ be a weakly convergent subsequence of $\{u_\varepsilon(\cdot)\}_{\varepsilon>0}$ with weak limit $u^*(\cdot)$. Then $\{u_{\varepsilon_k}(\cdot)\}_{k\geqslant 1}$ is bounded in $\mathcal{U}[t,T]=L^2_{\mathbb{F}}(t,T;\mathbb{R}^m)$. For any $u(\cdot)\in\mathcal{U}[t,T]$, we have

$$J(t,x;u(\cdot))+\varepsilon_k\mathbb{E}\int_t^T|u(s)|^2ds\geqslant V_{\varepsilon_k}(t,x)=J(t,x;u_{\varepsilon_k}(\cdot))+\varepsilon_k\mathbb{E}\int_t^T|u_{\varepsilon_k}(s)|^2ds. \tag{6.12}$$

Note that $u(\cdot)\mapsto J(t,x;u(\cdot))$ is sequentially weakly lower semi-continuous. Letting $k\to\infty$ in (6.12), we obtain

$$J(t,x;u^*(\cdot))\leqslant\varliminf_{k\to\infty}J(t,x;u_{\varepsilon_k}(\cdot))\leqslant J(t,x;u(\cdot)),\qquad\forall u(\cdot)\in\mathcal{U}[t,T].$$

Hence, $u^*(\cdot)$ is an open-loop optimal control of Problem (SLQ) at (t,x). $\square$

From the proof of Theorem 6.2, we see that the open-loop solvability of Problem (SLQ) at (t,x) is also equivalent the L^2-boundedness of $\{u_\varepsilon(\cdot)\}_{\varepsilon>0}$. In particular, the open-loop solvability of Problem (SLQ)^0 at (t,x) is equivalent the L^2-boundedness of $\{\Theta_\varepsilon(\cdot)X_\varepsilon(\cdot)\}_{\varepsilon>0}$ with $X_\varepsilon(\cdot)$ being the solution of

$$\begin{cases}dX_\varepsilon(s)=(A+B\Theta_\varepsilon)X_\varepsilon ds+(C+D\Theta_\varepsilon)X_\varepsilon dW(s),\qquad s\in[t,T],\\ X_\varepsilon(t)=x.\end{cases} \tag{6.13}$$

Since the $L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))$-norm of $X_\varepsilon(\cdot)$ is dominated by the L^2-norm of $\Theta_\varepsilon(\cdot)$, we conjecture that the L^2-boundedness of $\{\Theta_\varepsilon(\cdot)\}_{\varepsilon>0}$ will lead to the open-loop solvability of Problem (SLQ)^0 at (t,x). Actually, we have the following result.

Proposition 6.4. *Let* (H1)–(H2) *hold. Suppose* $u(\cdot)\mapsto J^0(0,0;u(\cdot))$ *is convex, and let* $\{\Theta_\varepsilon(\cdot)\}_{\varepsilon>0}$ *be the sequence defined by* (6.3). *If*

$$\sup_{\varepsilon>0}\int_0^T|\Theta_\varepsilon(s)|^2ds<\infty, \tag{6.14}$$

then the Riccati equation (4.7) *admits a regular solution* $P(\cdot)\in C([0,T];\mathbb{S}^n)$. *Consequently, Problem* (SLQ)^0 *is closed-loop solvable.*

Proof. For any $x\in\mathbb{R}^n$ and $\varepsilon>0$, let $X_\varepsilon(\cdot)$ be the solution of

$$\begin{cases}dX_\varepsilon(s)=\big[A(s)+B(s)\Theta_\varepsilon(s)\big]X_\varepsilon(s)ds+\big[C(s)+D(s)\Theta_\varepsilon(s)\big]X_\varepsilon(s)dW(s),\qquad s\in[0,T],\\ X_\varepsilon(0)=x.\end{cases} \tag{6.15}$$

By Itô's formula, we have

$$\begin{aligned}\mathbb{E}|X_\varepsilon(t)|^2&=|x|^2+\mathbb{E}\int_0^t\Big\{\big|[C(s)+D(s)\Theta_\varepsilon(s)]X_\varepsilon(s)\big|^2+2\big\langle[A(s)+B(s)\Theta_\varepsilon(s)]X_\varepsilon(s),X_\varepsilon(s)\big\rangle\Big\}ds\\&\leqslant|x|^2+\int_0^t\Big[\big|C(s)+D(s)\Theta_\varepsilon(s)\big|^2+2\big|A(s)+B(s)\Theta_\varepsilon(s)\big|\Big]\mathbb{E}|X_\varepsilon(s)|^2ds,\qquad\forall t\in[0,T].\end{aligned}$$

Thus, by Gronwall's inequality,

$$\begin{aligned}\mathbb{E}|X_\varepsilon(t)|^2&\leqslant|x|^2\exp\left\{\int_0^T\Big[|C(s)+D(s)\Theta_\varepsilon(s)|^2+2|A(s)+B(s)\Theta_\varepsilon(s)|\Big]ds\right\}\\&\leqslant|x|^2\exp\left\{K\left(1+\int_0^T|\Theta_\varepsilon(s)|^2ds\right)\right\},\qquad\forall t\in[0,T],\end{aligned} \tag{6.16}$$

where $K > 0$ is some constant depending only on $A(\cdot), B(\cdot), C(\cdot), D(\cdot)$. Hence,

$$
\begin{aligned}
\mathbb{E}\int_0^T |\Theta_\varepsilon(s)X_\varepsilon(s)|^2 ds &\leqslant \int_0^T |\Theta_\varepsilon(s)|^2 \mathbb{E}|X_\varepsilon(s)|^2 ds \\
&\leqslant |x|^2 \exp\left\{K\left(1+\int_0^T |\Theta_\varepsilon(s)|^2 ds\right)\right\}\int_0^T |\Theta_\varepsilon(s)|^2 ds,
\end{aligned} \tag{6.17}
$$

which, together with (6.14), implies the L^2-boundedness of $\{\Theta_\varepsilon(s)X_\varepsilon(s)\}_{\varepsilon>0}$. Thus, by Theorem 6.2, Problem $(\text{SLQ})^0$ is open-loop solvable at $t=0$, and by Theorem 5.3, Problem $(\text{SLQ})^0$ is finite. Now let $P:[0,T]\to\mathbb{S}^n$ such that (5.2) holds. Then by Theorem 5.3,

$$
R + D^\top PD \geqslant 0, \qquad \text{a.e.} \tag{6.18}
$$

Let $\{\Theta_{\varepsilon_k}(\cdot)\}$ be a weakly convergent subsequence of $\{\Theta_\varepsilon(\cdot)\}$ with weak limit $\Theta(\cdot)$. Since

$$
R + \varepsilon I + D^\top P_\varepsilon D \to R + D^\top PD \qquad \text{as} \quad \varepsilon\to 0
$$

and $\{R(\cdot)+\varepsilon I + D(\cdot)^\top P_\varepsilon(\cdot)D(\cdot)\}_{0<\varepsilon\leqslant 1}$ is uniformly bounded, we have

$$
B^\top P_{\varepsilon_k} + D^\top P_{\varepsilon_k} C + S = -(R+\varepsilon_k I + D^\top P_{\varepsilon_k} D)\Theta_{\varepsilon_k} \to -(R+D^\top PD)\Theta \qquad \text{weakly in } L^2.
$$

Also, note that

$$
B^\top P_{\varepsilon_k} + D^\top P_{\varepsilon_k} C + S \to B^\top P + D^\top PC + S \qquad \text{strongly in } L^2.
$$

Thus,

$$
-(R+D^\top PD)\Theta = B^\top P + D^\top PC + S.
$$

This implies

$$
\mathcal{R}\big(B^\top P + D^\top PC + S\big) \subseteq \mathcal{R}\big(R + D^\top PD\big), \qquad \text{a.e.} \tag{6.19}
$$

Since

$$
(R+D^\top PD)^\dagger (B^\top P + D^\top PC + S) = -\big(R+D^\top PD\big)^\dagger \big(R+D^\top PD\big)\Theta,
$$

and $\big(R+D^\top PD\big)^\dagger\big(R+D^\top PD\big)$ is an orthogonal projection, we have

$$
(R+D^\top PD)^\dagger (B^\top P + D^\top PC + S) \in L^2(0,T;\mathbb{R}^{m\times n}), \tag{6.20}
$$

and

$$
\Theta = -(R+D^\top PD)^\dagger (B^\top P + D^\top PC + S) + \big[I - (R+D^\top PD)^\dagger (R+D^\top PD)\big]\Pi
$$

for some $\Pi(\cdot)\in L^2(0,T;\mathbb{R}^{m\times n})$. Finally, letting $k\to\infty$, we have

$$
\begin{aligned}
P(t) = \lim_{k\to\infty} P_{\varepsilon_k}(t) &= G + \lim_{k\to\infty}\int_t^T \Big[P_{\varepsilon_k}A + A^\top P_{\varepsilon_k} + C^\top P_{\varepsilon_k} C + Q + (P_{\varepsilon_k}B + C^\top P_{\varepsilon_k} D + S^\top)\Theta_{\varepsilon_k}\Big] ds \\
&= G + \int_t^T \Big[PA + A^\top P + C^\top PC + Q + (PB + C^\top PD + S^\top)\Theta\Big] ds \\
&= G + \int_t^T \Big[PA + A^\top P + C^\top PC + Q - (PB + C^\top PD + S^\top)(R+D^\top PD)^\dagger (B^\top P + D^\top PC + S)\Big] ds,
\end{aligned}
$$

which, together with (6.18)–(6.20), implies that $P(\cdot)$ is a regular solution of (4.7). $\square$

7 An example

In this section we re-exam Example 2.2 to illustrate some results we obtained. In this example, the stochastic LQ problem admits a *continuous* open-loop optimal control at all $(t,x)\in[0,T)\times\mathbb{R}^n$, hence it is open-loop solvable, while the value function is *not* continuous in t; the corresponding Riccati equation has a unique solution $P(\cdot)$, which does *not* satisfy the range condition (4.8) and therefore is not regular. Therefore, the problem is *not* closed-loop solvable on any $[t,T]$. This example also tells us that the necessity part in Theorem 4.2 does not hold.

Recall the following Problem $(\text{SLQ})^0$ with one-dimensional state equation:

$$\begin{cases} dX(s)=\big[u_1(s)+u_2(s)\big]ds+\big[u_1(s)-u_2(s)\big]dW(s), \qquad s\in[t,1], \\ X(t)=x, \end{cases} \tag{7.1}$$

and cost functional

$$J^0(t,x;u(\cdot))=\mathbb{E}X(1)^2. \tag{7.2}$$

In this example, $u(\cdot)=(u_1(\cdot),u_2(\cdot))^\top$ and

$$\begin{cases} A=0, \quad B=(1,1), \quad C=0, \quad D=(1,-1), \\ G=1, \quad Q=0, \quad S=(0,0)^\top, \quad R=\begin{pmatrix}0&0\\0&0\end{pmatrix}. \end{cases}$$

The corresponding Riccati equation reads

$$\begin{cases} \dot P=P^2(1,1)\begin{pmatrix}P&-P\\-P&P\end{pmatrix}^\dagger\begin{pmatrix}1\\1\end{pmatrix}=\dfrac{P}{4}(1,1)\begin{pmatrix}1&-1\\-1&1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=0, \\ P(1)=1. \end{cases} \tag{7.3}$$

Obviously, (7.3) has a unique solution $P(\cdot)\equiv 1$, and

$$\begin{aligned} &\mathcal{R}\big(B(s)^\top P(s)+D(s)^\top P(s)C(s)+S(s)\big)=\mathcal{R}\big((1,1)^\top\big)=\big\{(a,a)^\top : a\in\mathbb{R}\big\}, \\ &\mathcal{R}\big(R(s)+D(s)^\top P(s)D(s)\big)=\mathcal{R}\left(\begin{pmatrix}1&-1\\-1&1\end{pmatrix}\right)=\big\{(a,-a)^\top : a\in\mathbb{R}\big\}. \end{aligned}$$

Thus, the range condition (4.8) does not hold and hence $P(\cdot)$ is not regular. By Theorem 4.4, the problem is not closed-loop solvable on any $[t,1]$.

Now for any $\varepsilon>0$, consider state equation (7.1) and the cost functional

$$J^0_\varepsilon(t,x;u(\cdot))=\mathbb{E}\left[X(1)^2+\varepsilon\int_t^T|u(s)|^2ds\right]. \tag{7.4}$$

The Riccati equation for the above problem reads

$$\begin{cases} \dot P_\varepsilon=P_\varepsilon^2(1,1)\begin{pmatrix}\varepsilon+P_\varepsilon&-P_\varepsilon\\-P_\varepsilon&\varepsilon+P_\varepsilon\end{pmatrix}^{-1}\begin{pmatrix}1\\1\end{pmatrix}=\dfrac{2}{\varepsilon}P_\varepsilon^2, \\ P_\varepsilon(1)=1, \end{cases} \tag{7.5}$$

whose solution is given by

$$P_\varepsilon(t)=\frac{\varepsilon}{\varepsilon+2-2t},\qquad t\in[0,1]. \tag{7.6}$$

Letting $\varepsilon\to 0$, we have

$$P_0(t)\triangleq\lim_{\varepsilon\to 0}P_\varepsilon(t)=\begin{cases}0, & 0\leqslant t<1,\\ 1, & t=1.\end{cases} \tag{7.7}$$

Thus, by Theorem 5.3, the original Problem (SLQ)^0 is finite with value function

$$V^0(t,x)=0,\quad 0\leqslant t<1;\qquad V^0(1,x)=x^2,\qquad \forall x\in\mathbb{R}. \tag{7.8}$$

Next, set

$$\Theta_\varepsilon\triangleq-(R+\varepsilon I+D^\top P_\varepsilon D)^{-1}(B^\top P_\varepsilon+D^\top P_\varepsilon C+S)=-\frac{P_\varepsilon}{\varepsilon}\begin{pmatrix}1\\1\end{pmatrix}.$$

Then the solution of

$$\begin{cases}dX_\varepsilon(s)=\big[A(s)+B(s)\Theta_\varepsilon(s)\big]X_\varepsilon(s)ds+\big[C(s)+D(s)\Theta_\varepsilon(s)\big]X_\varepsilon(s)dW(s)\\ \qquad\qquad =-\dfrac{2P_\varepsilon}{\varepsilon}X_\varepsilon(s)ds,\qquad s\in[t,T],\\ X_\varepsilon(t)=x\end{cases} \tag{7.9}$$

is given by

$$X_\varepsilon(s)=x\exp\left\{-\int_t^s\frac{2P_\varepsilon(r)}{\varepsilon}dr\right\}=\frac{\varepsilon+2-2s}{\varepsilon+2-2t}x,\qquad t\leqslant s\leqslant 1, \tag{7.10}$$

and hence

$$u_\varepsilon(s)\triangleq\Theta_\varepsilon(s)X_\varepsilon(s)=-\left(\frac{x}{\varepsilon+2-2t},\frac{x}{\varepsilon+2-2t}\right)^\top,\qquad t\leqslant s\leqslant 1. \tag{7.11}$$

Note that for $t\in[0,1)$,

$$u_\varepsilon(\cdot)\to-\left(\frac{x}{2-2t},\frac{x}{2-2t}\right)^\top\quad\text{in } L^2 \text{ as } \varepsilon\to 0.$$

Thus, by Theorem 6.2, the original Problem (SLQ)^0 is open-loop solvable at any $(t,x)\in[0,T)\times\mathbb{R}$ with an open-loop optimal control

$$u^*_{(t,x)}(s)=-\left(\frac{x}{2-2t},\frac{x}{2-2t}\right)^\top,\qquad t\leqslant s\leqslant 1, \tag{7.12}$$

which is continuous in $s\in[t,1]$.

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